Dendrites and groups acting on them III

Bruno Duchesne (Institut Elie Cartan, Nancy, France)

Into the forest summer school, Technion 2019

Last time, we have seen

- A Tits Alternative

Last time, we have seen

- A Tits Alternative
- Contraction properties that imply strong proximality.

Last time, we have seen

- A Tits Alternative
- Contraction properties that imply strong proximality.
- For a dendrite X, Homeo(X) is a Polish group.

Last time, we have seen

- A Tits Alternative
- Contraction properties that imply strong proximality.
- For a dendrite X, $\mathrm{Homeo}(X)$ is a Polish group.
- If branch points are dense, $\mathrm{Homeo}(X) \to \mathrm{Sym}(\mathrm{Br}(X))$ is an embedding.

Last time, we have seen

- A Tits Alternative
- Contraction properties that imply strong proximality.
- For a dendrite X, $\text{Homeo}(X)$ is a Polish group.
- If branch points are dense, $\text{Homeo}(X) \to \text{Sym}(\text{Br}(X))$ is an embedding.
- The two topologies coincide.

Last time, we have seen

- A Tits Alternative
- Contraction properties that imply strong proximality.
- For a dendrite X, Homeo(X) is a Polish group.
- If branch points are dense, Homeo(X) $\to$ Sym(Br(X)) is an embedding.
- The two topologies coincide.
- If $S \subset \{3, 4, \ldots, \infty\}$ is finite, $F, F' \subset D_S$ finite and $< F > \simeq < F' >$ then $\exists g \in$ Homeo(D_S) such g extends the graph isomorphism.

Oligomorphy

Definition

A group action $G \curvearrowright X$ *is oligomorphic if for any* $n \in \mathbf{N}$*, The action diagonal action* $G \curvearrowright X^n$ *has finitely many orbits.*

Theorem

If S *is finite then the action of* $\mathrm{Homeo}(D_S)$ *on* D_S *is oligomorphic.*

Property (OB)

Definition

A group has property (OB)

Property (OB)

Definition

A group has property (OB) if any isometric action on metric space has bounded orbits.

Property (OB)

Definition

A group has property (OB) if any isometric action on metric space has bounded orbits.

Remark

Property (OB) implies property (FA), (FH) for example.

Property (OB)

Definition

A group has property (OB) if any isometric action on metric space has bounded orbits.

Remark

Property (OB) implies property (FA), (FH) for example.

Remark

By a theorem of Struble, any locally compact second countable group has a proper left invariant metric.

Property (OB)

Definition

A group has property (OB) if any isometric action on metric space has bounded orbits.

Remark

Property (OB) implies property (FA), (FH) for example.

Remark

By a theorem of Struble, any locally compact second countable group has a proper left invariant metric.

Theorem (D.-Monod)

The group $\mathrm{Homeo}(D_S)$ *has property (OB).*

Property (OB)

Definition

A group has property (OB) if any isometric action on metric space has bounded orbits.

Remark

Property (OB) implies property (FA), (FH) for example.

Remark

By a theorem of Struble, any locally compact second countable group has a proper left invariant metric.

Theorem (D.-Monod)

The group $\mathrm{Homeo}(D_S)$ *has property (OB). Moreover, if* S *is finite it has property (T) with its Polish topology.*

Some simple dendrite groups

Theorem (D.-Monod)

The groups $\mathrm{Homeo}(D_S)$ *are simple*

Some simple dendrite groups

Theorem (D.-Monod)

The groups Homeo(D_S) *are simple and pairwise non-isomorphic.*

Steps of the proof of simplicity

1. Homeo(D_S) is generated by stabilizers of points of a given order.

Steps of the proof of simplicity

1. Homeo(D_S) is generated by stabilizers of points of a given order.
2. Homeo(D_S) is generated by pointwise stabilizers of branches around branch points.

Steps of the proof of simplicity

1. $\mathrm{Homeo}(D_S)$ is generated by stabilizers of points of a given order.
2. $\mathrm{Homeo}(D_S)$ is generated by pointwise stabilizers of branches around branch points.
3. If $1 \neq N \lhd \mathrm{Homeo}(D_S)$ then N is dendrominimal.

Steps of the proof of simplicity

1. $\mathrm{Homeo}(D_S)$ is generated by stabilizers of points of a given order.
2. $\mathrm{Homeo}(D_S)$ is generated by pointwise stabilizers of branches around branch points.
3. If $1 \neq N \lhd \mathrm{Homeo}(D_S)$ then N is dendrominimal.
4. N has some austro-boreal element n.

Steps of the proof of simplicity

1. $\mathrm{Homeo}(D_S)$ is generated by stabilizers of points of a given order.
2. $\mathrm{Homeo}(D_S)$ is generated by pointwise stabilizers of branches around branch points.
3. If $1 \neq N \lhd \mathrm{Homeo}(D_S)$ then N is dendrominimal.
4. N has some austro-boreal element n.
5. Let C be a branch. We may assume the austro-boreal arc I is in C.

Steps of the proof of simplicity

1. Homeo(D_S) is generated by stabilizers of points of a given order.
2. Homeo(D_S) is generated by pointwise stabilizers of branches around branch points.
3. If $1 \neq N \lhd \text{Homeo}(D_S)$ then N is dendrominimal.
4. N has some austro-boreal element n.
5. Let C be a branch. We may assume the austro-boreal arc I is in C.
6. Constructing homeomorphisms by patching.

Steps of the proof of simplicity

1. Homeo(D_S) is generated by stabilizers of points of a given order.
2. Homeo(D_S) is generated by pointwise stabilizers of branches around branch points.
3. If $1 \neq N \lhd \text{Homeo}(D_S)$ then N is dendrominimal.
4. N has some austro-boreal element n.
5. Let C be a branch. We may assume the austro-boreal arc I is in C.
6. Constructing homeomorphisms by patching.
7. If g fixes pointwise C then $g = [n, h]$ for some well-chosen h.

Ideas to prove the groups are non-isomorphic

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

1. Stabilizers of points are maximal subgroups.

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

1. Stabilizers of points are maximal subgroups.
2. $$\mathrm{Stab}(x) \simeq \left(\prod_{C \in \hat{x}} \mathrm{Homeo}(C) \right) \rtimes \mathrm{Sym}(\hat{x}).$$

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

1. Stabilizers of points are maximal subgroups.
2. $$\mathrm{Stab}(x) \simeq \left(\prod_{C \in \hat{x}} \mathrm{Homeo}(C) \right) \rtimes \mathrm{Sym}(\hat{x}).$$
3. An isomorphism $\mathrm{Homeo}(D_S) \to \mathrm{Homeo}(D_{S'})$ maps stabilizers of points to stabilizers of points with the same order.

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

1. Stabilizers of points are maximal subgroups.
2. $$\mathrm{Stab}(x) \simeq \left(\prod_{C \in \hat{x}} \mathrm{Homeo}(C) \right) \rtimes \mathrm{Sym}(\hat{x}).$$
3. An isomorphism $\mathrm{Homeo}(D_S) \to \mathrm{Homeo}(D_{S'})$ maps stabilizers of points to stabilizers of points with the same order.
4. This induces a map $\varphi \colon D_S \to D_{S'}$.

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

1. Stabilizers of points are maximal subgroups.
2. $$\mathrm{Stab}(x) \simeq \left(\prod_{C \in \hat{x}} \mathrm{Homeo}(C) \right) \rtimes \mathrm{Sym}(\hat{x}).$$
3. An isomorphism $\mathrm{Homeo}(D_S) \to \mathrm{Homeo}(D_{S'})$ maps stabilizers of points to stabilizers of points with the same order.
4. This induces a map $\varphi\colon D_S \to D_{S'}$.
5. It commutes with the center maps

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

1. Stabilizers of points are maximal subgroups.
2. $$\mathrm{Stab}(x) \simeq \left(\prod_{C \in \hat{x}} \mathrm{Homeo}(C) \right) \rtimes \mathrm{Sym}(\hat{x}).$$
3. An isomorphism $\mathrm{Homeo}(D_S) \to \mathrm{Homeo}(D_{S'})$ maps stabilizers of points to stabilizers of points with the same order.
4. This induces a map $\varphi \colon D_S \to D_{S'}$.
5. It commutes with the center maps
6. Thus it is continuous.

Ideas to prove the groups are non-isomorphic

Idea: Recover the dendrite from algebraic data.

1. Stabilizers of points are maximal subgroups.
2. $$\mathrm{Stab}(x) \simeq \left(\prod_{C \in \hat{x}} \mathrm{Homeo}(C) \right) \rtimes \mathrm{Sym}(\hat{x}).$$
3. An isomorphism $\mathrm{Homeo}(D_S) \to \mathrm{Homeo}(D_{S'})$ maps stabilizers of points to stabilizers of points with the same order.
4. This induces a map $\varphi \colon D_S \to D_{S'}$.
5. It commutes with the center maps
6. Thus it is continuous.
7. φ is a homeomorphism.

Definition

Let G be a Polish group. An element $g \in G$ is **generic** *if its conjugacy class is comeager.*

Genericity

Definition

Let G be a Polish group. An element $g \in G$ is **generic** *if its conjugacy class is comeager.*

Proposition (D.)

The Polish group $\text{Homeo}(D_S)$ *has a dense orbit if and only if* $S = \{\infty\}$

Theorem

The Polish group $\text{Homeo}(D_\infty)$ *has a comeager conjugacy class.*

Question

Is there a simple description of the generic elements ?

Kaleidoscopic groups

Burger-Mozes universal groups for regular trees.

Let T_d be the d-regular tree with $d \geq 3$.

Burger-Mozes universal groups for regular trees.

Let T_d be the d-regular tree with $d \geq 3$. Let $H \leq \mathrm{Aut}(T_d)$ acting transitively on vertices. The local action at x is the permutation group given by the action of H_x on edges attached to x.

Burger-Mozes universal groups for regular trees.

Let T_d be the d-regular tree with $d \geq 3$. Let $H \leq \text{Aut}(T_d)$ acting transitively on vertices. The local action at x is the permutation group given by the action of H_x on edges attached to x.

For any $\Gamma \leq \text{Sym}([d])$, there is a "universal" group $U_d(\Gamma)$ with local action given by Γ.

Colorings

Let $x \in \mathrm{Br}(D_n)$, we denote by $\hat{x}$ the set $\pi_0 (D_n \setminus \{x\})$.

Colorings

Let $x \in \mathrm{Br}(D_n)$, we denote by $\hat{x}$ the set $\pi_0 (D_n \setminus \{x\})$.

Definition

A **coloring** *of D_n is a map*

$$c\colon \sqcup_{x \in \mathrm{Br}(D_n)} \hat{x} \to [n]$$

such that $c|_{\hat{x}}\colon \hat{x} \to [n]$ is a bijection.

Colorings

Let $x \in \mathrm{Br}(D_n)$, we denote by $\hat{x}$ the set $\pi_0\left(D_n \setminus \{x\}\right)$.

Definition

A **coloring** *of D_n is a map*

$$c\colon \sqcup_{x \in \mathrm{Br}(D_n)} \hat{x} \to [n]$$

such that $c|_{\hat{x}}\colon \hat{x} \to [n]$ is a bijection. A coloring is **kaleidoscopic** *if*

Colorings

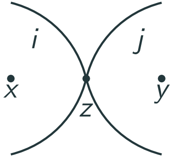
Let $x \in \mathrm{Br}(D_n)$, we denote by $\hat{x}$ the set $\pi_0(D_n \setminus \{x\})$.

Definition

A **coloring** *of D_n is a map*

$$c \colon \sqcup_{x \in \mathrm{Br}(D_n)} \hat{x} \to [n]$$

such that $c|_{\hat{x}} \colon \hat{x} \to [n]$ is a bijection. A coloring is **kaleidoscopic** *if for any $x \neq y \in \mathrm{Br}(D_n)$ and $i \neq j \in [n]$, there is $z \in]x, y[$ such that $c(U_z(x)) = i$ and $c(U_z(y)) = j$.*



Proposition (D.-Monod-Wesolek)

For any n, the set of kaleidoscopic colorings is a dense G_δ in the space of all colorings of D_n.

Uniqueness of kaleidoscopic colorings

Theorem (D.-Monod-Wesolek)

Let X, Y be dendrites homeomorphic to D_n and c and d be kaleidoscopic colorings of X and Y, respectively. Then there exists a homeomorphism $h\colon X \to Y$ such that $d \circ h = c$.

Uniqueness of kaleidoscopic colorings

Theorem (D.-Monod-Wesolek)

Let X, Y *be dendrites homeomorphic to* D_n *and* c *and* d *be kaleidoscopic colorings of* X *and* Y*, respectively. Then there exists a homeomorphism* $h\colon X \to Y$ *such that* $d \circ h = c$*.*

Furthermore, let $e_0, e_1 \in X$ *be distinct end points and likewise* $f_0, f_1 \in Y$*. Let* $x \in [e_0, e_1]$ *and* $y \in [f_0, f_1]$ *be branch points with* $c_x(e_i) = d_y(f_i)$ *for* $i = 0, 1$*. Then* h *can be chosen such that* $h(e_i) = f_i$ *for* $i = 0, 1$ *and such that* $h(x) = y$*.*

Local action

Definition

Let c be a coloring of D_n. The **local action** *of $g \in \mathrm{Homeo}(D_n)$ at $x \in \mathrm{Br}(D_n)$ is the element $\sigma_c(g, x)$ of $\mathrm{Sym}([n])$ defined by the cocycle*

$$\sigma_c\colon\ \mathrm{Homeo}(D_n)\times\mathrm{Br}(D_n) \longrightarrow \mathrm{Sym}([n]), \quad \sigma_c(g, x) := c_{g(x)}\circ g\circ c_x^{-1}$$

Definition

Let c be a coloring of D_n. For any permutation group $\Gamma \leq \mathrm{Sym}(n)$, the group with local action Γ is defined to be

$$\mathcal{K}_c(\Gamma) = \{g \in \mathrm{Homeo}(D_n) : \forall x \in \mathrm{Br}(D_n),\ \sigma_c(g,x) \in \Gamma\}.$$

Kaleidoscopic groups

Definition

Let c be a coloring of D_n. For any permutation group $\Gamma \leq \mathrm{Sym}(n)$, the group with local action Γ is defined to be

$$\mathcal{K}_c(\Gamma) = \{g \in \mathrm{Homeo}(D_n) : \forall x \in \mathrm{Br}(D_n),\ \sigma_c(g, x) \in \Gamma\}.$$

When c is a kaleidoscopic coloring, we call $\mathcal{K}_c(\Gamma)$ a **kaleidoscopic group** *with local action Γ.*

Examples of Kaleidoscopic groups

- $\mathcal{K}(\{1\})$ is the subgroup of G_n that preserves the coloring c.

Examples of Kaleidoscopic groups

- $\mathcal{K}(\{1\})$ is the subgroup of G_n that preserves the coloring c.
- $\mathcal{K}(\mathrm{Sym}([n]))$ is simply G_n itself.

Main properties of Kaleidoscopic groups

Theorem (D.-Monod-Wesolek)

(i) *The abstract group* $\mathcal{K}(\Gamma)$ *is simple and uniformly perfect.*

Main properties of Kaleidoscopic groups

Theorem (D.-Monod-Wesolek)

(i) *The abstract group* $\mathcal{K}(\Gamma)$ *is simple and uniformly perfect.*

(ii) *The permutation group* $\mathcal{K}(\Gamma)$ *is always primitive; it is doubly transitive if and only if* Γ *is transitive.*

Main properties of Kaleidoscopic groups

Theorem (D.-Monod-Wesolek)

(i) *The abstract group $\mathcal{K}(\Gamma)$ is simple and uniformly perfect.*

(ii) *The permutation group $\mathcal{K}(\Gamma)$ is always primitive; it is doubly transitive if and only if Γ is transitive.*

(iii) *The permutation group $\mathcal{K}(\Gamma)$ is never doubly primitive: its point-stabilizers admit a system of imprimitivity isomorphic to Γ and decompose as permutational wreath product over Γ.*

Main properties of Kaleidoscopic groups

Theorem (D.-Monod-Wesolek)

(i) *The abstract group* $\mathcal{K}(\Gamma)$ *is simple and uniformly perfect.*

(ii) *The permutation group* $\mathcal{K}(\Gamma)$ *is always primitive; it is doubly transitive if and only if* Γ *is transitive.*

(iii) *The permutation group* $\mathcal{K}(\Gamma)$ *is never doubly primitive: its point-stabilizers admit a system of imprimitivity isomorphic to* Γ *and decompose as permutational wreath product over* Γ*.*