EXERCISES ON CUBE COMPLEXES

Selected exercises from the notes

The notes can be found at `https://www.wescac.net/into_the_forest.pdf`. The exercises in the text ask you to fill in details in various places. Here are some of them:

(1) Let X be a CAT(0) cube complex. Let $Y \subset X$ be a convex subcomplex. Prove that Y is again a CAT(0) cube complex.
(2) Show that if $y \in X^{(0)}$, then the set of combinatorial halfspaces containing y is a consistent orientation.
(3) Let $\Lambda \subset \Gamma$ be median-convex. Show that Λ is convex in the metric sense: any geodesic of Γ with endpoints in Λ lies in Λ. Also prove the converse.
(4) Let Γ be a median graph and let Λ be a median-convex subgraph. Suppose that $x, y \in \Gamma$ are adjacent. Show that $\mathfrak{g}_\Lambda(x), \mathfrak{g}_\Lambda(y)$ are adjacent or equal. Using this, extend the gate map over edges to get a 1–lipschitz retraction $\mathfrak{g}_\Lambda : \Gamma \to \Lambda$.
(5) Let Γ be a median graph and let e be an edge. Let u be a vertex of e and let $\overleftarrow{e}$ be the preimage of u under the gate map $\Gamma \to e$. Prove that $\overleftarrow{e}$ is median-convex.
(6) Let X be a CAT(0) cube complex. In the lectures, we constructed, given $x, y, z \in X^{(0)}$, a candidate median for x, y, z. Prove that it is unique (i.e. it is the unique vertex m with $\mathsf{d}_1(x, y) = \mathsf{d}_1(x, m) + \mathsf{d}_1(m, y)$ and similarly for the pairs x, z and y, z).
(7) Let $Y \subset X$ be a subcomplex of the CAT(0) cube complex X, with $Y^{(1)}$ median-convex. Let $x \in X$ be a vertex and let H be a hyperplane. Then H separates x from Y if and only if H separates x from $\mathfrak{g}_Y(x)$, where $\mathfrak{g}_Y : X^{(1)} \to Y^{(1)}$ is the gate map.
(8) Let X be a CAT(0) cube complex and let Y, Z be disjoint median-convex subgraphs of $X^{(1)}$. Suppose that $Y^{(0)} \sqcup Z^{(0)} = X^{(0)}$. Show that there exists an edge e such that $Y^{(0)} = \overleftarrow{e}$ and $Z^{(0)} = \overrightarrow{e}$.

More exercises

(1) Tile a closed surface S of genus $g \geq 2$ by squares, so that links (which are all circles) have length at least 4. This gives a nonpositively curved square complex X homeomorphic to S. Do this in such a way that (1) no hyperplane in X crosses itself; (2) more generally, if e, f are edges dual to the same hyperplane, then e, f don't have a common vertex. How many squares do you need, in terms of g?
(2) Let X be a CAT(0) cube complex, and let H, H' be crossing hyperplanes. Let $\overleftarrow{H}, \overleftarrow{H'}$ be associated (necessarily intersecting) halfspaces. Form a new cube complex Y by passing to the largest subcomplex of $X - \overleftarrow{H} \cap \overleftarrow{H'}$. Show that Y is CAT(0).
(3) Let X be a CAT(0) cube complex and let $\mathcal{H}$ be a subset of the hyperplanes of X. Show that the CAT(0) cube complex Y dual to the wallspace $(X^{(0)}, \mathcal{H})$ is a quotient of X that can be realised topologically by collapsing each $\mathcal{N}(H), H \in \mathcal{H}$, which we view as $H \times [-\frac{1}{2}, \frac{1}{2}]$: fix a contraction of $[-\frac{1}{2}, \frac{1}{2}]$ to a point and perform it fibrewise to collapse $\mathcal{N}(H)$. Show that for any convex subcomplex Z of Y, the preimage of Z under $X \to Y$ is convex, and that the preimage of any hyperplane is a hyperplane.
(4) Let γ be a 1–skeleton geodesic in a D–dimensional CAT(0) cube complex, and let $\ell \geq 0$. Suppose that $|\gamma| \geq R(\ell, D+1)$. Show that γ is crossed by at least ℓ disjoint hyperplanes. (Here $R(\ell, D+1)$ denotes the Ramsey number associated to $\ell, D+1$.)

(5) Let X be a CAT(0) cube complex. Prove that if $\dim X < \infty$, then (X, d_2) is quasi-isometric to $(X^{(1)}, \mathsf{d}_1)$.
(6) Let X be a CAT(0) cube complex. Prove that $X^{(1)}$ is hyperbolic if and only if the following holds. There exists a constant k such that whenever there is an embedding $[0,p] \times [0,q] \to X$ which is an isometric embedding on 1–skeleta, we have $\min\{p,q\} \leq k$. (Here $p,q \in \mathbb{N}$ an $[0,p]$ and $[0,q]$ are given the obvious 1–dimensional cubical structures.) Hint: since $X^{(1)}$ is median, to prove hyperbolicity, you need to show that if γ, γ' are geodesics with common endpoints, then they are (uniformly) Hausdorff-close.
(7) Let X be a CAT(0) cube complex and let $\mathcal{H}$ be the set of hyperplanes. Suppose that there is a finite set F and a map $c : \mathcal{H} \to F$ such that, for all $f \in F$, the hyperplanes in $c^{-1}(f)$ are all disjoint. Prove that $X^{(1)}$ embeds isometrically in the product of $|F|$ trees.
(8) Prove the following fact, mentioned earlier in the notes: let X be a CAT(0) cube complex and let $\mathcal{H}$ be the set of hyperplanes. Suppose we can write $\mathcal{H} = \mathcal{A} \sqcup \mathcal{B}$, where every hyperplane in $\mathcal{A}$ crosses every hyperplane in $\mathcal{B}$. Then $X \cong A \times B$, where A, B are CAT(0) cube complexes. Moreover, each hyperplane in $\mathcal{A}$ has the form $H \times B$, where H is a hyperplane of A (and a similar description holds for $\mathcal{B}$).
(9) Let X be a CAT(0) cube complex with $|X^{(0)}| < \infty$. Let G be a group acting on X by cubical automorphisms. Prove that G fixes a point in X. Deduce that, if Y is a proper CAT(0) cube complex on which the group G acts properly and cocompactly by cubical automorphisms, then G contains finitely many conjugacy classes of finite subgroups.