DENDRITES AND GROUPS ACTING ON THEM: EXERCISES

INTO THE FOREST SUMMER SCHOOL, TECHNION 2019

Exercise 1: Branches are open. Let X be a dendron. Prove that branches (i.e. connected components of the complement of a point) are open.

Exercise 2: Dendrons and dendrites. Prove that a dendron is a dendrite if and only if it is separable. You may show that some countable collections of branches is a basis for the topology and use the particular case of Uryshon's metrization theorem: A compact hausdorff space is metrizable if and only if it is second countable.

Exercise 3: Convexity and connectedness. Let X be a dendrite and $C \subseteq X$. Prove that C is connected if and only if it is convex (i.e. $\forall x, y \in C$, $[x, y] \subseteq C$).

Exercise 4: Continuity of the center map. Let X be dendrite. Prove that the center map $c\colon X^3 \to X$ is continuous.

Exercise 5: No free arc. A *free arc* is an arc (not reduced to a point) whose interior is open in X, equivalently this interior does not meet $\mathrm{Br}(X)$. For any dendrite X not reduced to a point, show that the following conditions are equivalent:

(i) X has no free arc.
(ii) $\mathrm{Br}(X)$ is dense in X.
(iii) $\mathrm{Ends}(X)$ is dense in X.

Exercise 6: Ends points are generic points. Show that in a dendrite X without free arcs, $\mathrm{Ends}(X)$ is comeager (i.e. it is a dense countable intersection of open subsets).

Exercise 7: The fixed point property for dendrites. Prove that any homeomorphism of a dendrite has a fixed point (many direct or indirect proofs are possible).

Exercise 8: Convex hull and end points. Let X be a dendrite and $M \subseteq X$ be a closed subspace. Show that $Ends([M]) \subseteq M$ where $[M]$ is the smallest subdendrite containing M (its closed convex hull).

Exercise 9: Actions of products are elementary. Let $G_1 \times G_2$ be a group product acting on a dendrite. Prove that at least one of the factors acts elementarily. Hints: Assume that G_1 acts non-elementarily, consider the unique G_1-invariant subdendrite and show this is an arc or a point.

Exercise 10: Strong proximality. Let X be a dendrite and G be a group acting dendro-minimally.

(i) Show that for any two branches around non-end points U_1, U_2 then there is $g \in G$ such that $g(U_1) \subseteq U_2$.
(ii) Show that if moreover the action is non-elementary then there are uncountably many end points.
(iii) Prove that a non-elementary dendro-minimal action is strongly proximal.

Exercise 11: Bijections preserving the center map. Let X be a dendrite without free arcs. Let $f\colon \mathrm{Br}(X) \to \mathrm{Br}(X)$ be a bijection. Prove that f is the restriction of some homeomorphism of X if and only if f commutes with the center map: $f(c(x,y,z)) = c(f(x), f(y), f(z))$ for all branch points x, y, z.

Exercise 12: Topologies. Let X be a dendrite without free arcs. Prove that the compact-open topology (i.e. the uniform convergence topology) coincides with the topology of pointwise convergence topology on the set of branch points.

Exercise 13: Maximal subgroups. Let $G_S = \mathrm{Homeo}(D_S)$ where $S \subseteq \{3, 4, \ldots, \infty\}$ and D_S is the associated Ważewski dendrite. Prove that stabilizers of points are maximal subgroups.