Topological Groups

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Abstract

These are lecture notes for the course "Topics in Topological Groups" taught in Fall 2019. The notes are heavily based on Terry Tao's book "Hilbert's Fifth Problems and Related Topics", and do not make any substantial modification of this text, except for a slight reorganization.

Contents

1 Introduction

1.1 The Basics

1.1.1 Basic definitions and examples

Definition 1.1. A topological group is a group G with a topology such that the multiplication function $\cdot : G \times G \to G$ and the inverse function $^{-1} : G \to G$ are continuous.

Example 1.2. Examples of topological groups include:

1. Any group with the discrete/trivial topology
2. $(\mathbb{R}, +)$, $(\mathbb{R}/\mathbb{Z}, +)$, $\mathrm{GL}(n, \mathbb{R}), \ldots$ with their standard topology. (These are groups which are also smooth manifolds with smooth operations. Such groups are called *Lie groups*)
3. Subgroup and quotients of topological groups are topological groups.
4. For a prime p, the p-adic integers $\mathbb{Z}_p$ with their topology. Recall that on $\mathbb{Z}$ we can define the following norm

$$|a|_p = \begin{cases} 0 & \text{if } a = 0 \\ p^{-k} & \text{if } a = mp^k \text{ and } \gcd(m, p) = 1 \end{cases}.$$

   The p-adic integers $\mathbb{Z}_p$ is the metric completion of $\mathbb{Z}$ with respect to the metric $d_p(a, b) = |a - b|_p$.

   Exercise 1.3. Show that the addition and multiplication operations of $\mathbb{Z}$ extend continuously to $\mathbb{Z}_p$ and the $(\mathbb{Z}_p, +)$ is a topological group.

5. $(\mathcal{H}, +)$ for a Hilbert space $\mathcal{H}$ with its metric or weak topologies.

Example 1.4. Non-examples of topological groups include:

1. Infinite groups with the cofinite topology. Recall that this is the topology in which the closed sets are the finite subsets and the whole set.
2. $(\mathbb{R}_\ell, +)$ where $\mathbb{R}_\ell$ is $\mathbb{R}$ with the topology generated by the half-closed intervals $\{[a, b) : a < b\}$.

In this course we will focus on *locally compact* topological groups.

Definition 1.5. A topological space is *locally compact* if every point has a compact neighborhood. [1]

Exercise 1.6. Which of the above examples is locally compact?

Another important example of a group topology is the following:

[1] By neighborhood we always mean a subset which contains a (given) point in its interior.

Definition 1.7. Let G be a group, the *profinite topology* on G is the topology generated by the basis consisting of all left cosets of all finite index subgroups of G.

Exercise 1.8. Show that this is indeed a basis for a group topology, and that the basis sets are clopen (i.e. closed and open)

Theorem 1.9 (Euclid). *There are infinitely many prime numbers.*

Here is the proof of this theorem by Fürstenberg using the profinite topology on $\mathbb{Z}$.

Proof. Assume that there are finitely many primes, $p_1, \dots, p_n$. Then, by definition

$$\mathbb{Z} - \{\pm 1\} = p_1\mathbb{Z} \cup \dots \cup p_n\mathbb{Z}.$$

By the exercise above each of $p_i\mathbb{Z}$ is closed, and thus $\{\pm 1\}$ is open, contradicting the fact that all basis sets are infinite. □

1.1.2 Basic properties

Lemma 1.10. *Let G be a topological group and let $g \in G$. The following functions are homeomorphisms:*

- *the inverse function $^{-1} : G \to G$;*
- *the left (resp. right) multiplication map $L_g : G \to G$ (resp. $R_g : G \to G$) defined by $L_g(h) = gh$ (resp. $R_g(h) = hg^{-1}$);*
- *and the conjugation map $c_g : G \to G$ defined by $c_g(h) = ghg^{-1}$.*

Proof. They are all continuous by definition and the maps $^{-1}$, $L_{g^{-1}}$, ($R_{g^{-1}}$,) $c_{g^{-1}}$ are their respective continuous inverses. □

Exercise 1.11. The topology of a topological group is fully defined by its identity neighborhood basis.

1.2 Inverse Limits

An important example of topological groups comes from the construction of inverse limits.

Definition 1.12. A *directed set* $(I, \leq)$ is a poset such that for all $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$.

An *inverse system of (topological) groups* $\mathcal{G} = (\{G_i\}_{i \in I}, \{f_{ij}\}_{i \leq j \in I})$ is a set of groups G_i indexed by a directed set $(I, \leq)$, and a set of (continuous) surjective homorphisms $f_{ij} : G_j \to G_i$ for all $i \leq j$, such that:

1. $f_{ii} = \mathrm{id}$ for all $i \in I$
2. $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k \in I$.

The *inverse limit* (or *projective limit*) of $\mathcal{G}$ is the group

$$\varprojlim G_i = \left\{ (g_i)_{i\in I} \in \prod_{i\in I} G_i \;\middle|\; \forall i \le j, f_{ij}(g_j) = g_i \right\}$$

The inverse limit is endowed with the subset topology of the product space $\prod_{i\in I} G_i$

It is often the case that the directed set is simply $(\mathbb{N}, \le)$, in which case it suffices to describe the maps $f_{n,n+1} : G_{n+1} \to G_n$.

Example 1.13.

- Infinite products can be seen as inverse limits of finite products. For example, $\prod G_n = \varprojlim G_1 \times \ldots \times G_n$ here the index set is $\mathbb{N}$ and the nth group is $G_1 \times \ldots \times G_n$, with the obvious projection maps between them.
- A more interesting example is the p-adic integers. Again, indexed over $\mathbb{N}$, consider $G_n = \mathbb{Z}/p^n\mathbb{Z}$, let f_{nm} be the obvious quotient map $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$.

  Exercise 1.14. Prove $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

- The *Solenoid* is the following inverse limit: Consider $G_n = \mathbb{R}/p^n\mathbb{Z}$, and the natural maps (alternatively, think of G_n as $\mathbb{R}/\mathbb{Z}$ and the maps $f_{n,n+1}(x) = p \cdot x$).

  Exercise 1.15. Show that $(\mathbb{Z}_p \times \mathbb{R})/\Delta\mathbb{Z} = \varprojlim G_n$ where $\Delta\mathbb{Z} = \{(n,n) \mid n \in \mathbb{Z}\}$.

- Consider $G_n = \mathbb{Z}/n\mathbb{Z}$ indexed by the directed set $(\mathbb{N}, |)$ (where $|$ is divisibility)

  Exercise 1.16. What is $\varprojlim G_n$? Is the image of $\mathbb{Z} \to \varprojlim G_n$ dense? (Hint: Chinese Remainder Theorem)

Exercise 1.17 (Pro-finite completion)**.** Let G be a residually-finite group (i.e. $\forall 1 \ne g \in G$ there exists a finite quotient $\phi : G \to Q$ such that $\phi(g) \ne 1$). Form the inverse limit $\varprojlim G/N$ ranging over all finite index subgroups of G. Show that $G \to \varprojlim G/N$ is injective with dense image. This inverse limit is called the *profinite completion of G*

Exercise 1.18 (The universal property of $\varprojlim G_i$)**.** Show that $\varprojlim G_i$ and its projection maps $f_i : \varprojlim G_i \to G_i$ satisfy the following universal property: For every (topological) group G, and maps $\phi_i : G \to G_i$ such that for all $i \le j$, $\phi_i = f_{ij} \circ \phi_j$. There exists a unique (continuous) homomorphism $\phi : G \to \varprojlim G_i$ such that $\phi_i = f_i \circ \phi$ for all $i \in I$.

Exercise 1.19. Show that if the group G_i are locally compact, and the kernels of the maps f_{ij} are compact, then $\varprojlim G_i$ is locally compact.

Lemma 1.20 (A criterion for inverse limit). *Let G be a locally compact Hausdorff topological group, let $(\mathcal{N}, \leq)$ be a directed set of normal compact subgroups of G, such that $N \leq N'$ if $N \geq N'$, and such that for all identity neighborhood U there exists $N \in \mathcal{N}$ such that $N \subseteq U$, then $G \simeq \varprojlim_{N \in \mathcal{N}} G/N$.*

Proof. The morphisms $G \to G/N$ induce a morphism $\phi : G \to \varprojlim_{N \in \mathcal{N}} G/N$. It remains to show that this map is a homeomorphism - i.e it remains to show that it is an open bijection.

Since G is Hausdorff, for every $1 \neq g \in G$, there exists an open identity neighborhood U such that $g \notin U$. We conclude that there exists $N \subseteq U$ and $g \notin N$, so $\phi(g) \neq \phi(1)$. This shows that ϕ is injective.

Now, consider an identity neighborhood $U_0/N_0 \subset G/N_0$. That is, U_0 is an identity neighborhood in G such that $U_0 N_0 = U_0$.

Claim. *$\phi(U)$ is dense in $(\varprojlim_{N \in \mathcal{N}} G/N) \cap \pi_{N_0}^{-1}(U_0/N_0)$.*

To see this, let

$$V = (\varprojlim_{N \in \mathcal{N}} G/N) \cap \pi_N^{-1}(U_0/N_0) \cap \pi_1^{-1}(U_1/N_1) \cap \ldots \cap \pi_k^{-1}(U_k/N_k)$$

be a non-empty open set in $(\varprojlim_{N \in \mathcal{N}} G/N) \cap \pi_{N_0}^{-1}(U_0/N_0)$. Let $N' \in \mathcal{N}$ be a compact normal subgroup such that $N' \leq N_0 \cap N_1 \cap \ldots \cap N_k$. Since V is non-empty there exists $gN' \in G/N'$ such that $\phi_{N_i N'} \in U_i/N_i$ for all $i = 0, 1, \ldots, k$. This means that $gN' \cap U_0 \cap U_1 \ldots \cap U_k \neq \varnothing$. But since $U_i N' = U_i$ for all $i = 0, 1, \ldots, k$, we get $g \in U_0 \cap U_1 \ldots \cap U_k$, and hence $\phi(g) \in V$.

From the claim we get in particular that $\phi(G)$ is dense in $\varprojlim_{N \in \mathcal{N}} G/N$.

But also, we can deduce that ϕ is a local homeomorphism: if U_0 is a compact identity neighborhood and $N_0 \in \mathcal{N}$ is such that $N_0 \subseteq U_0$, we may assume without loss of generality that $U_0 = U_0 N_0$ (both U_0 and N_0 are compact, and thus so is there product). Then $\phi(U_0)$ is dense in $V := (\varprojlim_{N \in \mathcal{N}} G/N) \cap \pi_{N_0}^{-1}(U_0/N_0)$. Since U_0 is compact, and V is Hausdorff (this will be explained later), and the map $\phi|_{U_0}$ is an injective continuous map with dense image in V, we deduce that $\phi|_{U_0} : U_0 \to V$ is a homeomorphism. Therefore, we get a homeomorphism $\phi|_{U'} : U' \to V'$ where U', V' are open identity neighborhoods (by restricting $\phi|_{U_0}$ to the preimage of the interior of V).

In particular, the image of $\phi(G)$ contains an open set, hence $\phi(G)$ is open, and hence closed. But since $\phi(G)$ is dense, we get that ϕ is onto.

Now, ϕ is open, as every U can be seen as $\bigcup_{g \in U} gU_g$ where $U_g \subseteq U'$ is an open identity neighborhood. By the homomorphism property,

$$\phi(U) = \phi(\bigcup_{g \in U} gU_g) = \bigcup_{g \in U} \phi(g)\phi(U_g)$$

and by the above discussion $\phi(U_g)$ are open, and thus so are $\phi(g)\phi(U_g)$. □

1.3 Course objective

In the next Chapters we will establish the following remarkable structure theorem for a locally compact group G.

Theorem 1.21 (Gleason-Yamabe)**.** *Let G be a locally compact topological group. Then G has an open subgroup G' such that for all open identity neighborhood $U \subseteq G$, there exists a normal closed subgroup $N \lhd G'$ contained in U, such that G'/N is a Lie group.*

In particular, every locally compact Hausdorff group has an open subgroup which is an inverse limit of Lie groups.

The structure of Lie groups is a well-studied area of math, and should be the focus of an independent course. However, note that for that theorem we include under Lie groups all discrete groups.

Example 1.22. Let $G = \mathbb{Q}_p \rtimes_\phi \mathbb{Z}$ where $\phi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Q}_p)$ is given by $\phi(k)x = p^k x$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$. Let $U = \mathbb{Z}_p \times 1 \leq G$ be an open identity neighborhood. Then G does not contain a non-trivial normal subgroup in U, as the conjugation by the subgroup $1 \times \mathbb{Z}$ will eventually move every non-trivial element of U outside U. This shows that one has to pass to an open subgroup $G' \leq G$, for example $G' = \mathbb{Z}_p$ which is an inverse limit.

In particular, we will be able to deduce the following solution of Hilbert's Fifth Problem. A *topological manifold* is a Hasudorff space which is locally homeomorphic to $\mathbb{R}^n$. A group is *locally Euclidean* if it is a topological manifold.

Theorem 1.23 (Montgomery-Zippin,Gleason)**.** *If G is a locally Euclidean topological group then G is a Lie group.*

On the way, we will see many fundamental theorems about topological group, which are of great importance on their own right – such as the existence and uniqueness of the Haar measure, Peter-Weyl Theorem, van Danzig Lemma, ...

2 Building-Up Separation

2.1 Separation axioms (and uniform structure)

We recall the following separation axioms:

Definition 2.1. A topological space X is

T_0 if for every $x, y \in X$ there exists an open set U such that $y \notin U \ni x$ or $x \notin U \ni y$.

T_1 if for every $x, y \in X$ there exists an open set U such that $y \notin U \ni x$. (or alternatively, if every point is closed)

T_2 **or Hausdorff** if for every $x, y \in X$ there exist disjoint open sets U, V such that $x \in U, y \in V$.

T_3 **or (Hausdorff) regular** if it is Hausdorff and for every $x \in X$ and a closed set $x \notin C \subseteq X$ there exist open sets U, V such that $x \in U, C \subseteq V$.

$T_{3^1/_2}$ **or Tychonoff or completely regular** if it is Hausdorff and for every $x \in X$ and a closed set $x \notin C \subseteq X$ there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(C) = 0$.

T_4 **or (Hausdorff) normal** if it is Hausdorff and for every disjoint $C, D \subseteq X$ there exists disjoint open sets U, V such that $C \subseteq U, D \subseteq V$.

Remark 2.2. Note that $T_i \implies T_j$ for all $j \geq i$.

The additional structure of being a topological groups enables us to go up the separation scale.

Proposition 2.3. *A T_0 topological group G is $T_{3^1/_2}$.*

Proof. Let G be a T_0 topological group. First let us show that G is T_1. For all $x, y \in G$ the homeomorphism $L_y \circ \cdot^{-1} \circ L_{x^{-1}}$ exchanges x, y. Hence, $T_0 \implies T_1$.

To prove that G is Hausdorff, it suffices to prove that one can separate 1 from any $1 \neq g \in G$ by disjoint open sets. By T_1, $U := G - \{g\}$ is an open identity neighborhood. By continuity of $\cdot : G \times G \to G$, there exists an open identity neighborhood $1 \in V$ such that $V^2 \subseteq U$. By replacing V with $V \cap V^{-1}$ we may further assume that $V = V^{-1}$. Note that $1 \in V, g \in gV$ are open neighborhoods and $V \cap gV = \varnothing$ as otherwise $h = gh'$ for $h, h' \in V \implies g = h(h')^{-1} \in V^2 \subseteq U$ contradicting the assumption on U.

The same proof can be used to show that T_3. Setting $U = G - C$, finding $1 \in V$ such that $V^2 \subseteq U$ and $V = V^{-1}$, and taking the open neighborhoods $1 \in V$ and $C \subseteq CV$.

To show that G is $T_{3^1/_2}$, following the proof of Urysohn's Lemma it suffices to find identity neighborhoods $\{U_q\}_{q \in \mathbb{Q} \cap [0,1]}$, such that for all $q < q' \in \mathbb{Q} \cap [0,1]$, we have $\overline{U_q} \subseteq U_{q'} \subseteq G - C$. (Then define $f : G \to [0,1]$ by

$$f(g) = \inf(\{1\} \cup \{q : g \in U_q\})$$

and show that it is continuous by following the proof of Urysohn's Lemma.) We do so by induction on an enumeration of $\mathbb{Q} \cap [0,1] = \{1, 0, 1/2, \ldots\}$, making sure at each step, for all $q < q'$ previously constructed $U_q V \subseteq U_q'$ for some open symmetric identity neighborhood $1 \in V \subseteq G$. Start by setting $U_1 = G - C$. Let V be a symmetric open identity neighborhood such that $V^2 \subseteq U_1$. Set $U_0 = V$. For the induction step, let q be the next number in the enumeration, let q' and q'' be the maximal (resp. minimal) previously defined, such that $q' < q$ (resp. $q < q''$). Let $U_{q'} V \subseteq U_{q''}$ for some open identity neighborhood V, and let W be an open identity neighborhood such that $W = W^{-1}$ and $W^2 \subseteq V$. Define $U_q = U_{q'} W$. Clearly, $U_q W \subseteq U_q''$. □

Remark 2.4. We cannot do better than $T_{3^1/_2}$. For example the uncountable product $\mathbb{R}^\mathbb{R}$ (i.e, the space of functions $\mathbb{R} \to \mathbb{R}$ with pointwise convergence) is a T_0 topological group which is not normal.

Later in this section we will prove that locally compact Hausdorff groups which are first countable are metrizable. In fact, they can be endowed with a left-invariant metric.

2.2 Uniform structure

In the proof of Proposition 2.3 we have used something weaker than the fact that G is a topological group.

Definition 2.5. A uniform space is a space X, together with collection Φ of subsets $U \subseteq X \times X$ such that:

- $U \in \Phi \implies \Delta X \subseteq U$
- $U \in \Phi, U \subseteq V \subseteq X \times X \implies V \in Phi$
- $U, V \in \Phi \implies U \cap V \in \Phi$
- $U \in \Phi \implies \exists V \in \Phi : V \circ V \subseteq U$ (where $A \circ B = \{(x,z) : \exists y : (x,y) \in A, (y,z) \in B\}$ for relations $A \subseteq X \times Y$ and $B \subseteq Y \times Z$).
- $U \in \Phi \implies U^{-1} \in \Phi$ (where $A^{-1} = \{(y,x) : (x,y) \in A\} \subseteq Y \times X$ for a relation $A \subseteq X \times Y$)

The uniform structure (X, Φ) defines a topology (X, τ) by:

$$V \in \tau \iff \forall x \in V, \exists U \in \Phi : U[x] \subseteq V$$

where $U[x] := \{y : (x,y) \in U\}$.

A topological group G has two uniform space structures. The left (resp. right) uniform structure is $U \in \Phi$ if it contains $\{(x,y) | x^{-1}y \in V\}$ (resp. $\{(x,y) | xy^{-1} \in V\}$) for some open identity neighborhood V.

Remark 2.6. Note that metric spaces and compact topological spaces have natural uniform structures.

Exercise 2.7. 1. Define *uniform continuity* for a function between uniform spaces.

2. Show that a function $f : X \to Y$ with compact support is uniformly continuous.

3. Show that the left (resp. right) multiplication L_g is uniformly continuous with respect to the left (resp. right) uniform structure.

2.3 Hausdorffization

Proposition 2.8. *For every topological group G, the subset $\overline{\{1\}}$ is a normal closed subgroup with trivial topology, and ${}^{G}\!/\!_{\overline{\{1\}}}$ is Hausdorff.*

Proof. First, note that $g\overline{\{1\}} = \overline{\{g\}} = \overline{\{1\}}$ for all $g \in \overline{\{1\}}$, (for the first equality note that the homeomorphism L_g maps $1 \mapsto g$; for the second equality note that $\overline{\{g\}} \subseteq \overline{\{1\}}$ by definition and $\supseteq$ follows from the homeomorphism exchanging $1, g$). Moreover, $\overline{\{1\}} = \overline{\{1\}}^{-1}$ since $\cdot^{-1}$ is a homeomorphism. This shows that $\overline{\{1\}}$ is a subgroup. Since conjugations are homeomorphisms fixing 1, it follows that $\overline{\{1\}}$ is normal. Finally, ${}^{G}\!/\!_{\overline{\{1\}}}$ is T_1 since $\overline{\{1\}}$ is clearly closed, and hence it is Hausdorff. □

Observation 2.9. The topology of G is the pullback of the topology of ${}^{G}\!/\!_{\overline{\{1\}}}$.

2.4 σ-compactness

Definition 2.10. A topological space is *σ-compact* if it can be exhausted by a sequence of compact sets.

Proposition 2.11. *Every locally compact topological group has an open subgroup which is σ-compact.*

Proof. Let V be an compact symmetric identity neighborhood, then $\langle V \rangle = \bigcup V^n$ is a σ-compact open subgroup. □

2.5 Metrizability

Definition 2.12. A topological space is *first countable* if every point has a countable local basis. That is, for all $x \in X$ there is a countable collection of neighborhoods of x, $\mathcal{U}_x = \{U_1, U_2 \ldots\}$, such that for every neighborhood $x \in V$ there exists $i \in \mathbb{N}$ such that $x \in U_i \subseteq V$.

Remark 2.13. 1. Note that metric spaces are first countable.

2. Note that for a topological group, it suffices to find such countable local basis for the identity element.

Theorem 2.14 (Birkhoff-Kakutani). *A topological group is metrizable if and only if it is Hasudorff and first countable. In fact, in such a case it has a left-invariant metric (that is, $d(x, y) = d(gx, gy)$ for all $x, y, g \in G$).*

Proof. The "only if" part is clear, we will prove the "if" part. Assume G is Hausdorff and first countable, then there is a countable local basis of open neighborhoods at 1, $\mathcal{U}_1 = \{U_n\}_{n\in\mathbb{N}}$. By $T_{3^1/_2}$, there are functions $f_n : G \to [0,1]$ such that $f_n(1) = 1$ and $f_n(G - U_n) = 0$. In fact, a closer look at the proof, shows that we can choose f_n to be uniformly continuous. Consider the function $f = \sum_n 2^{-n} f_n$. From the construction, f is uniformly continuous, $f(1) = 1$, and (from the Hausdorff assumption) $f(g) \neq 1$ for all $1 \neq g \in G$.

Consdier the metric $d_f : G \times G \to \mathbb{R}$ given by

$$d_f(g,h) = \sup_{x\in G} |f(g^{-1}x) - f(h^{-1}x)| = \sup_{x\in G} |f(g^{-1}hx) - f(x)|.$$

It satisfies the triangle inequality because it is the pull back of the sup norm. $d_f(g,h) = 0 \iff g = h$ since $f(x) = 1 \iff x = 1$, and it is continuous because f is uniformly continuous. In other words, the group topology is finer than the metric topology.

By construction $d(1,g) < 2^{-n} \implies f(g) > 1 - 2^{-n} \implies g \in U_n$ we get that the group topology is finer than the metric topology.

It is also apparent that the metric d_f is left-invariant. □

Lemma 2.15. *Let G be a locally compact topological group. Then there is an open subgroup G' of G such that, for every identity neighborhood U in G, there exists a compact $N \lhd G'$ contained in U, such that G'/N is first countable.*

Proof. Without loss of generality, we may assume that U is compact. We construct by induction a sequence U_n of identity neighborhoods such that $U_0 = U$, and $U_{n+1}^2 \subseteq U_n$ and $U_{n+1}^U \subseteq U_n$ (where $A^B = \{a^b : a \in A, b \in B\}$ and $a^b = b^{-1}ab$). We have seen how to get V such that $V^2 \subseteq U$. Using the continuity of the map $G \times G \to G$ given by $(a,b) \mapsto a^b$, we see that the set $O = \{(a,b) : a^b \in U_n\}$ is open, and contains $\{1\} \times U$, by compactness of U there exists an open identity neighborhood set V' such that $V \times U \subset O$. Set $U_{n+1} = V \cap V'$.

Set $N = \bigcap_n U_n$ and $G' = \langle U \rangle$.

Exercise 2.16. Show that $N \lhd G'$ and G'/N is first countable.

This completes the proof □

To sum it up, in the following sections we will often assume that our groups are locally compact. Therefore, by passing to an open subgroup and taking an arbitrarily small quotient (such as in the statement of Gleason-Yamabe), we may assume that the group if σ-compact, locally compact and metrizable.

3 Connected and Totally Disconnected Groups

3.1 The identity component

Definition 3.1. A topological space is *totally disconnected* if the connected components are singletons.

Proposition 3.2. *Let G be a topological group. Let us denote by G_0 the identity connected component. G_0 is a closed connected normal subgroup and ${}^{G}/_{G_0}$ is a totally disconnected topological group.*

Proof. G_0 is a normal subgroup: Note that $1 \in gG_0 \cup G_0$ is connected for all $g \in G_0$, and therefore $gG_0 \subseteq G_0$, so $G_0 G_0 \subseteq G_0$. The inverse map is a homeomorphism and therefore $G_0^{-1} = G_0$. Similarly, it is normal, since the conjugation maps are homeomorphisms.

The group ${}^{G}/_{G_0}$ is totally disconnected: Let A be a connected subset of ${}^{G}/_{G_0}$ containing 1. Let $B = \pi^{-1}(A)$ be its preimage. Then if U, V are open sets such that $1 \in B \cap U, B \cap V$ are disjoint. Then, for every $b \in B$ $bG_0 \subseteq B$ and either $bG_0 \subseteq U$ or V. This means that $\pi(U), \pi(V)$ are open[2] in ${}^{G}/_{G_0}$, and $\pi(U) \cap A, \pi(V) \cap A$ are disjoint. Since A is connected $\pi(U) \supseteq A$ and therefore $U \supseteq B$. This shows that B is connected, but then $B \subseteq G_0$, which implies that $A = \{1\}$. That is, all connected sets are singletons. □

This proposition shows that every topological group is "connected − by − totally-disconnected".

Observation 3.3. Every open subgroup is also closed. In particular, connected groups do not have proper open subgroups.

3.2 The van Dantzig Theorem

In view of the Gleason-Yamabe Theorem (Theorem 1.21), connected groups are inverse limits of Lie groups. Thus, it makes sense to focus for now on totally disconnected locally compact groups (these groups are often abbreviated as *tdlc groups*).

Theorem 3.4 (van Dantzig's Theorem)**.** *Every totally disconnected locally compact Hausdorff group has a local basis of compact open subgroups.*

Example 3.5. The group $\mathbb{Q}_p$ of p-adic rationals, is totally disconnected. It has a local basis of groups given by the compact open subgroups $\{p^n \mathbb{Z}_p\}_{n \in \mathbb{N}}$.

We will see more examples later.

To prove the theorem we will need the following lemma

Lemma 3.6. *Let X be a totally disconnected compact Hausdorff space, and let $x, y \in X$ be disjoint points. Then there exists a clopen subset containing x but not y.*

[2] note that for topological groups quotient maps are always open

Proof. Let K be the intersection of all clopen neighbrohoods of x. We want to show that $K = \{x\}$, for that it suffices to show that K is connected.

Assume that O is clopen in K containing x. Let $O' = K - O$. Both O, O' are closed in K, and K is closed in X, and hence O, O' are closed in X. X is normal (because it is compact and Hausdorff), therefore there are open disjoint sets U, U' containing O, O' respectively. There exists a clopen neighborhood C of x that is contained in $U \cup U'$ (otherwise by the finite intersection property K will contain a point outside $U \cup U'$). $C \cap U$ is clopen in C (as its complement is $C \cap U'$), and thus clopen neighborhood of x in X. It follows that $K \subseteq U$ and $K = O$, as desired. □

Proof of Theorem 3.4. Let U be a compact identity neighborhood. Applying the lemma to 1 and $y \in \partial U$ we can find an identity neighborhood V_y which is clopen in U and does not contain y. By compactness of ∂U, we can find an identity neighborhood V which is clopen in U and does not intersect ∂U. It follows that V is clopen in G.

By continuity of multiplication and compactness of V, there exists a symmetric open identity neighborhood W in V such that $WV \subseteq V$. It follows that $\langle W \rangle \subseteq V \subseteq U$ is an open (hence clopen) subgroup contained in U. □

As a corollary we get a special case of the Gleason-Yamabe Theorem for totally disconnected groups.

Corollary 3.7. *Let G be a totally disconnected locally compact Hausdorff group, then G has an open subgroup that is the inverse limit of discrete groups.*

Proof. By the van Dantzig Theorem, let G' be an open compact subgroup of G. Let U be some identity neighborhood of G'. By the van Dantzig Theorem, there exists an open subgroup $H \leq G'$ in U. Since G' is compact, H has finite index in G'. It follows that $N = \bigcap_{g \in G'} H^g$ is open (as a finite intersection of open groups) and normal. Hence ${}^{G'}\!/\!{}_{N}$ is discrete. The map $G' \to \varprojlim {}^{G}\!/\!{}_{N} \subseteq \prod_N {}^{G'}\!/\!{}_{N}$ shows that G' is an inverse limit of discrete groups. □

3.3 Profinite groups

Definition 3.8. A *profinite group* is a compact, Hausdorff, totally disconnected group. Equivalently, it is the inverse limit of finite groups.

Example 3.9.

- The profinite completion of a group. Recall that for a (residually finite) group G, the profinite completion of G is the inverse limit of ${}^{G}\!/\!{}_{N}$ for all finite index normal subgroups $N \lhd G$.
- $\mathbb{Z}_p, \mathrm{SL}(n, \mathbb{Z}_p)$.
- For a Galois field extension K/k of fields of characteristic 0. That is, K/k is algebraic (every element in K is a root of a polynomial over k), and for each irreducible polynomial f over k the number of roots is 0 or $\deg(f)$. The Galois group $\mathrm{Gal}(K/k)$ is the group of all field automorphisms of K

that are identity on k. For every intermediate field $K > L > k$, we may consider $\mathrm{Gal}(K/L) \le \mathrm{Gal}(K/k)$.

We can consider on $\mathrm{Gal}(K/k)$ the *Krull topology* which is generated by the identity neighborhoods $\mathrm{Gal}(K/L)$ where L ranges over all finite Galois extensions of k.

In fact, every profinite group can be made isomorphic to a Galois group.

An important recent theorem about profinite groups is the following:

Theorem 3.10 (Nikolov–Segal 2007). *If G is a topologically finitely generated profinite group then $H \le G$ is open if and only if H has finite index.*

The proof of this theorem uses the CFSG.

Corollary 3.11. *If G, H are profinite, and G is topologically finitely generated. Every surjective group homomorphism $G \to H$ is continuous.*

Proof. Indeed, every open subgroup of H is of finite index, and hence its preimage is open in G by the Nikolov-Segal Theorem. □

Remark 3.12. It is easy to see that a closed subgroup of finite index is open.

3.4 Locally finite graphs and the Cayley-Abels graph

3.4.1 Groups of automorphisms of graphs

Lemma 3.13. *Let Γ be a locally finite connected graph. Then $\mathrm{Aut}(\Gamma)$ the group of automorphisms of Γ, with the pointwise convergence topology (or equivalently the compact open topology) is a tdlc group.*

Proof. Clearly $\mathrm{Aut}(\Gamma)$ is totally disconnected, since it is a subspace of the totally disconnected space $V(\Gamma)^{V(\Gamma)}$. To prove that it is locally compact, consider the open identity neighborhood $1 \in \mathrm{Stab}_G(v_0)$ for some vertex $v_0 \in V(\Gamma)$. Let us denote by B_n the (vertices in the) ball of radius n around v_0 in Γ. Then U clearly preserves B_n, and so in fact $U \le \prod B_n^{B_n}$ which is compact by Tychonoff. □

The van Danzig Theorem takes a simply form in this case: $G = \mathrm{Aut}(\Gamma)$ has a local basis of open compact subgroups given by $\{\mathrm{Fix}_G(F)\}$ where F ranges over finite subsets of Λ and $\mathrm{Fix}_G(F)$ is the pointwise stabilizer of F in G.

Example 3.14. If Γ is the d-regular rooted tree. I.e. it is the tree that has one vertex of degree d called the root, and all other vertices of degree $d+1$. Then $\mathrm{Aut}(\Gamma)$ is a profinite group. Its finite quotients can be seen as the automorphism groups of the balls around the root. That is, it is the inverse limit of the wreath products $(\dots(S_d \wr_{[d]} S_d) \wr_{[d]} \dots) \wr_{[d]} S_d$ where S_d is the symmetric group on $[d] = \{1, \dots, d\}$, and $G \wr_X H$ is the wreath product of G and $H \curvearrowright X$, i.e $(\bigoplus_{x \in X} G) \rtimes H$ where H acts on $\bigoplus_{h \in H} G$ by permuting its coordinates according to the action $H \curvearrowright X$.

3.4.2 The Cayley-Abels graph

Let G be a compactly generated, locally compact, totally disconnected group. Let U be a compact open subgroup, and let S be a compact generating set, satisfying $S = USU$.

Definition 3.15. The *Cayley-Abels graph* is the graph $\Gamma = \mathrm{Cay}(G, S, U)$ whose vertex set $V(\Gamma) = G/U$ and edges are (gU, gsU) for all $s \in S$ and $g \in G$.

Note that Γ has finite valency $d := |USU : U|$, and that G acts on Γ vertex transitively. The action of G might not be faithful. This happens when $\mathrm{Core}_G(U) = \bigcap U^g \neq 1$.

The action $G \to \mathrm{Aut}(\Gamma)$ is continuous, since $\mathrm{Stab}(gU) = gUg^{-1}$ are open.

Finally, one can get new groups acting on a finite valence tree from this construction by noting that T_d the d-regular tree covers $\Gamma = \mathrm{Cay}(G, U, S)$ by the covering map $p : T_d \to \Gamma$. We can consider $\tilde{G}$ to be the group of automorphisms of T_d that cover elements of g, that is, all elements $\tilde{g} \in \mathrm{Aut}(T_d)$ such that there exists $g \in G$ such that $p \circ \tilde{g} = g \circ p$. In particular, $\pi_1(\Gamma) \leq \tilde{G}$ as Deck transformations. $\tilde{G}$ fits into the following short exact sequence:

$$1 \to \pi_1(\Gamma) \to \tilde{G} \to G \to 1.$$

Moreover, $\tilde{G}$ is closed in $\mathrm{Aut}(T_d)$.

4 The Haar Measure

4.1 Radon measures and the Haar measure

The (left) Haar measure is a (nice) measure on a topological group G that is invariant under (left) multiplication. One instance of this measure is the Lebesgue measure on $\mathbb{R}^n$.

Definition 4.1 (Haar measure)**.** A *Radon measure* Let X be a σ-compact locally compact Hausdorff topological space. A *Radon measure* is a Borel measure (σ-additive positive measure on the Borel σ-algebra $\mathcal{B}(X)$ of X) such that:

1. $\mu(K) < \infty$ for all compact $K \subseteq X$
2. $\mu(E) = \sup\{\mu(K) \mid K \subseteq E \text{ compact}\}$ for all $E \in \mathcal{B}(X)$
3. $\mu(E) = \inf\{\mu(U) \mid U \supseteq E \text{ open}\}$ for all $E \in \mathcal{B}(X)$

A *left (resp. right) Haar measure* on a σ-compact locally compact Hausdorff topological group is a non-zero Radon measure which is invariant under left (resp. right) multiplication. That is, $\mu(E) = \mu(gE)$ for all $E \in \mathcal{B}(G)$, or equivalently, $\int f(gx)d\mu = \int f(x)d\mu$

Example 4.2.

- The Lebesgue measure on $(\mathbb{R}^n, +)$.
- The counting measure on a discrete group.
- The Haar measure on $\mathbb{Z}_p$ can be discribed in the following way. Every number in $\mathbb{Z}_p$ can be written uniquely as $\sum_n a_n p^n$ for $0 \le a_n < p$. This gives a (continuous) map $\prod_n \{0, \dots, p-1\} \to \mathbb{Z}_p$. The Haar measure on $\mathbb{Z}_p$ is the push-forward of the product measure.

Exercise 4.3. Prove the claims in the previous example.

Exercise 4.4. Show that if U is open and non-empty in G and μ is a left Haar measure on G, then $\mu(U) > 0$.

Theorem 4.5. *Every locally compact (σ-compact, Hausdorff) topological group admits a unique Haar measure up to a scalar multiplication.*

4.2 Existence of the Haar measure.

To prove this theorem we recall the Riesz Representation Theorem which basically states that integration identifies Radon measures and positive functionals on compactly supported continuous functions.

Definition 4.6. Let $C_c(X)$ be the space of complex valued continuous functions with compact support on X. A *positive* functional on $C_c(X)$ is a linear functional $I : C_c(X) \to \mathbb{R}$ such that $I(f) \ge 0$ for all $f \ge 0$ in $C_c(X)$.

Theorem 4.7 (Riesz Representation Theorem)**.** *The map that maps $\mu \mapsto I_\mu$, where $I_\mu(f) = \int f(x)d\mu(x)$, is a bijection between Radon measures on X and positive linear functionals.*

We will not prove this theorem in this course, the proof of this theorem can be found in any measure theory textbook.

Proof of Existence of Haar measure in Theorem 4.5. Let us restate the existence of the Haar measure in Theorem 4.5 in view of Theorem 4.7. We want to show that there exists a non-trivial positive bounded functional satisfying $I(f) = I(\tau(g)f)$ where τ is the left translation action of G on $C_c(G)$ $\tau(g)f(x) = f(g^{-1}x)$ for all $x, g \in G$, and $f \in C_c(G)$.

In fact it suffices to consider the space $C_c(G)^+$ of non-negative functions in $C_c(G)$, and to consider functions $I : C_c(G)^+ \to [0,\infty)$ which are additive, homogeneous (with respect to positive scalars), and invariant under left translations.

To prove the existence of such a functional we will use a compactness argument. For the sake of the argument let us fix some non-zero compactly supported function $f_0 : G \to [0,\infty)$, this function will serve as our gauge; we will require $I(f_0) = 1$. For all $\epsilon > 0$ and functions $f_1, \dots, f_n \in C_c(G)^+$ we will find a function $I = I_{\epsilon,f_0,\dots,f_n} : C_c(G)^+ \to [0,\infty)$ satisfying:

1. (normalization) $I(f_0) = 1$
2. (homogeneity) $I(\lambda f) = \lambda f$ for all $\lambda > 0, f \in C_c(G)^+$,
3. (approximate-additivity) $|I(f_i + f_j) - I(f_i) - I(f_j)| \le \epsilon$ for all $0 \le i, j \le n$,
4. (invariance) $I(\tau(g)f) = I(f)$ for all $g \in G, f \in C_c(G)^+$,
5. (uniform bound) and $I(f) \le M(f)$ for all $f \in C_c(G)^+$, where $M : C_c(G)^+ \to [0,\infty)$ is independent of $\epsilon, f_1, \dots, f_n$.

Assuming we have constructed such functions, let us show how to use them to construct the desired functional. Given $\epsilon, f_0, \dots, f_n$, let us denote by $\mathcal{I}_{\epsilon,f_0,\dots,f_n}$ the set of all such functionals. These sets are closed subsets of $\prod_{f \in C_c(G)^+}[0, M(f)]$ which is compact by the Tychonoff Theorem. The existence of $I_{\epsilon,f_0,\dots,f_n}$ shows that these sets satisfy the finite intersection property. Therefore, by compactness there exists a function $I \in \bigcap \mathcal{I}_{\epsilon,f_0,\dots,f_n}$, which must be a left invariant positive functional, as desired.

To prove the existence of the function $I = I_{\epsilon,f_0,\dots,f_n}$, let $\delta > 0$ be a small number to be determined later. By uniform continuity of $f_1, \dots, f_n$, let U be an identity neighborhood such that $|f_i(x') - f_i(x)| \le \delta$ for all $x \in G, x' \in xU$ and for all $1 \le i \le n$. Finally, let $\psi : G \to [0,1]$ be a non-zero continuous function supported on U.

Define

$$[f : \psi] := \inf \left\{ \sum_{k=1}^{m} c_k \;\middle|\; \forall c_1, \dots, c_m > 0, g_1, \dots, g_m \in G \text{ such that } f \le \sum_{k=1}^{m} c_k \tau(g_k)\psi \right\},$$

and

$$I(f) = \frac{[f : \psi]}{[f_0 : \psi]}.$$

Clearly I is normalized, homogeneous and left-invariant. It is also easy to see that $[f : \psi] \leq [f : f_0][f_0 : \psi]$, from which the uniform bound follows by setting $I(f) \leq [f : f_0] =: M(f)$. It thus remains to show the approximate-additivity.

It is clear that $I(f_i + f_j) \leq I(f_i) + I(f_j)$. Therefore we have to show that $I(f_i) + I(f_j) \leq I(f_i + f_j) + \epsilon$. Let $c_k > 0, g_k \in G$ $(k = 1, \ldots, m)$ such that $f_i + f_j \leq \sum c_k \tau(g_k)\psi$. Set $c'_k = \frac{f_i(g_k)+\delta}{f_i(g_k)+f_j(g_k)+2\delta} c_k$ and $c''_k = \frac{f_j(g_k)+\delta}{f_i(g_k)+f_j(g_k)+2\delta} c_k$, so that we have $c_k = c'_k + c''_k$. Then, we claim that by choosing U small enough we have

$$f_i \leq \sum c'_k \tau(g_k)\psi + 4\delta$$

and

$$f_j \leq \sum c''_k \tau(g_k)\psi + 4\delta.$$

Evaluating the summands on the right hand side on some $x \in G$ we get

$$\begin{aligned} c'_k \tau(g_k)\psi(x) &= \frac{f_i(g_k) + \delta}{f_i(g_k) + f_j(g_k) + 2\delta} c_k \psi(g_k^{-1}x) \\ &\geq \frac{f_i(x)}{f_i(x) + f_j(x) + 4\delta} c_k \psi(g_k^{-1}x) \end{aligned}$$

where the second inequality follows since ψ supported on U, and thus it is non-zero only if $x \in g_k U$, and hence $|f_i(x) - f_i(g_k)| \leq \delta$ and $|f_j(x) - f_j(g_k)| \leq \delta$.

Hence,

$$\begin{aligned} \sum c'_k \tau(g_k)\psi(x) + 4\delta &\geq \frac{f_i(x)}{f_i(x) + f_j(x) + 4\delta} \sum c_k \tau(g_k)\psi(x) + 4\delta \\ &\geq \frac{f_i(x)}{f_i(x) + f_j(x) + 4\delta} (f_i(x) + f_j(x)) + 4\delta \\ &\geq f_i(x). \end{aligned}$$

Now, let $\phi \in C_c(X)$ be some function such that $\phi = 1$ on the support of f_i and f_j, then $f_i \leq \sum c'_k \tau(g_k)\psi + 4\delta\phi$ and $f_j \leq \sum c''_k \tau(g_k)\psi + 4\delta\phi$. By monotonicity of I,

$$I(f_i) + I(f_j) \leq \frac{1}{[f_0 : \psi]} \sum (c'_k + c''_k) + 8\delta I(\phi) \leq \frac{\sum c_k}{[f_0 : \psi]} + 8\delta M(\phi).$$

By taking δ small enough, and taking an infimum over all such c_k, we get the desired bound. □

4.3 Uniqueness of the Haar measure

To prove the uniqueness we will use the Radon-Nikodym derivative.

Definition 4.8. The measure μ is *absolutely continuous* with respect to ν, denoted $\mu \ll \nu$ if for all measurable E, $\nu(E) = 0 \implies \mu(E) = 0$.

Theorem 4.9 (Radon-Nikodym Derivative)**.** *If $\mu \ll \nu$ are σ-finite then there exists f measurable such that $d\mu = f d\nu$. This f is denoted as $\frac{d\mu}{d\nu}$ and is well-defined up to ν-null sets.*

Proof of Uniqueness of Haar measure (Theorem 4.5). Let μ, ν be two Haar measures. Then, $\mu + \nu$ is also a left Haar measure, and $\mu \ll \mu + \nu$. Therefore, for our proof we may assume without loss of generality that $\mu \ll \nu$.

By the Radon-Nikodym there exists a measurable $h : G \to (0, \infty)$ such that $d\mu = h \cdot d\nu$. Then, for all $f \in C_c(G)$ and all $z \in G$

$$\begin{aligned}\int f(x)h(x)d\nu(x) &= \int f(x)d\mu(x) \\ &= \int f(zx)d\mu(x) \\ &= \int f(zx)h(x)d\nu(x) \\ &= \int f(x)h(z^{-1}x)d\nu(x)\end{aligned}$$

where at two equalities we use the invariance of μ and ν. Hence, $\int f(x)(h(x) - h(z^{-1}x))d\nu(x)$ for all $f \in C_c(G)$. It follows that for all $z \in G$ $h(x) - h(z^{-1}x) = 0$ for ν-a.e. x.

By Fubini, for ν-a.e. x

$$0 = \int |h(x) - h(z^{-1}x)|d\mu(z) = \int |h(x) - h(z^{-1})|d\mu(z)$$

. We conclude that $h \equiv c$ is ν-a.e., for some $c > 0$, and $\mu = c\nu$ as desired. □

4.4 The modular function

Note that if μ is a left Haar measure on G, and $g \in G$, then $(R_g) * \mu$ (right translation of μ by g) is again a left Haar measure on G, therefore it is a scalar multiple of μ. The function $\Delta : G \to (0, \infty)$ such that $(R_g)_*\mu = \Delta(g)\mu$ is called the modular function. The modular function measures how much a left Haar measure is also a right Haar measure. We say that a group is *unimodular* if its left Haar measure is also a right Haar measure.

Exercise 4.10. Prove that the modular function is a continuous homomorphism, independent of μ. Deduce that if G is compact or perfect (i.e G is equal to its commutator subgroup) then it is unimodular.

Show that the push-forward of a left Haar measure under the inverse map is a right Haar measure, and show that it is equal to $\Delta(g)^{\pm} d\mu(g)$ for the right assignment of the sign $\pm$.

Exercise 4.11. Find a left Haar measure for $G = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \leq \mathrm{GL}(2, \mathbb{R})$ and show that it is not unimodular.

5 Compact Groups

In this section let G be a compact topological group, and let μ be its (bi-invariant) Haar probability measure on G. Our goal will be to find many finite dimensional representations of G, in the sense of the following theorem. As a corollary we will deduce the Gleason-Yamabe Theorem for compact groups.

Theorem 5.1 (Baby Peter-Weyl Theorem)**.** *Let G be a compact Hausdorff group, and let $1 \neq g \in G$, then there exists a finite dimensional representation $G \to \mathrm{GL}(V)$ in which g acts non-trivially.*

5.1 The left regular representation and convolution

Denote by $L^2(G) = L^2(G, \mu)$ the space of square-integrable functions $f : G \to \mathbb{C}$ (up to μ-everywhere equivalence). This space is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)}\, d\mu(x).$$

On this space there is an action by G given by $\tau(y)f(x) = f(y^{-1}x)$ for all $y \in G, f \in L^2(G)$. The operators $\tau(y)$ are unitary, as it is easy to verify $\tau(y)^{-1} = \tau(y^{-1}) = \tau(y)^*$. This action is the *regular representation of* G.

The strategy for finding finite dimensional representations of G is to find finite dimensional subspaces of $L^2(G)$ which are invariant under the regular action. To do so, we will find an operator that commutes with the left regular representation and study its eigenspaces. These operators are coming from the convolution on $L^2(G)$.

The *convolution* of $f, g \in L^2(G)$ is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)d\mu(y)$$

Claim 5.2. *The convolution of functions in $L^2(G)$ is well-defined, continuous and in $L^2(G)$.*

Proof. It is well-defined since $f * g$ can also be written as the inner product of f and $\tilde{g}_x(y) = \overline{g(y^{-1}x)}$, which is again in $L^2(G)$ as the Haar measure is right invariant. The map $x \mapsto \tilde{g}_x$ is continuous, hence $f * g(x) = \langle f, \tilde{g}_x \rangle$ is continuous. Since G is compact, $f * g$ is automatically in $L^2(G)$. □

Remark 5.3. We are using the fact that G is compact in the previous claim. There are functions $f, g \in L^2(\mathbb{R})$ whose convolution is not in $L^2(\mathbb{R})$.

Given $g \in L^2(G)$, denote by $T_g : L^2(G) \to L^2(G)$ the operator given by $f \mapsto f * g$.

Claim 5.4. *The operator T_g satisfies:*

1. *(commutation) T_g commutes with the regular representation.*

2. *(self-adjoint)* T_g *is self-adjoint if* $g(x^{-1}) = \overline{g(x)}$ *for all* $x \in G$*.*

3. *(compact)* T_g *is a compact operator (i.e, it maps bounded sets to precompact sets).*

Proof. We have,

$$\begin{aligned}
\tau(y)T_g(f)(x) &= \int f(z)g(z^{-1}(y^{-1}x))d\mu(z) \\
[z' := yz] &= \int f(y^{-1}z')g((y^{-1}z')^{-1}y^{-1}x)d\mu(z') \\
&= \int f(y^{-1}z')g((z')^{-1}x)d\mu(z') \\
&= T_g\tau(y)f(x).
\end{aligned}$$

This shows that T_g commutes with the regular representation. In fact, one has $\tau(y)(f * g) = (\tau(y)f) * g = f * (\tau(y)g)$ for all $y \in G$.

If $g(x^{-1}) = \overline{g(x)}$ for all $x \in G$, then

$$\begin{aligned}
\langle f * g, h\rangle &= \int (f * g)(x)\overline{h(x)}d\mu(x) \\
&= \iint f(y)g(y^{-1}x)\overline{h(x)}d\mu(y)d\mu(x) \\
[g(z^{-1}) = \overline{g(z)}] &= \iint f(y)\overline{g(x^{-1}y)h(x)}d\mu(y)d\mu(x) \\
&= \langle f, h * g\rangle .
\end{aligned}$$

This shows that under this assumption on g, T_g is self-adjoint.

Compactness will follow from the following exercise by taking $K(x,y) := g(x^{-1}y)$. Note that since G is compact (and $\mu(G) = 1$),

$$\|K\|^2_{L^2(G^2)} = \iint |g(x^{-1}y)|^2 d\mu(x)d\mu(y) \tag{5.1}$$

$$= \iint |g(y)|^2 d\mu(x)d\mu(y) \tag{5.2}$$

$$= \int \|g\|_2^2 d\mu(x) = \|g\|^2_{L^2(G)} \tag{5.3}$$

□

Exercise 5.5. Let $K \in L^2(X \times Y, \mu \times \nu)$, and let $T : L^2(X,\mu) \to L^2(Y,\nu)$ be the operator defined by

$$T(f)(y) = \int_X K(x,y)f(x)d\mu(x).$$

Then T is compact. [Hint: 1. Approximate K by combinations of function $a(x)b(y)$ where $a(x) \in L^2(X), b(y) \in L^2(Y)$. 2. Show that T is a (norm) limit of finite rank bounded operators. 3. Show that such a limit must be compact.]

5.2 Spectral Theory and the Peter-Weyl Theorem

For compact self adjoint we have the following.

Theorem 5.6 (the spectral theorem of compact self-adjoint operators)**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there exists a countable sequence $\lambda_n \searrow 0$ of non-zero eigenvalues, such that $\mathcal{H} = V_0 \oplus \bigoplus_n V_{\lambda_n}$ where V_{λ_n} are finite dimensional λ_n-eigenspaces, and $V_0 = \ker(T)$.*

Now we are ready to prove Theorem 5.1. In fact, we will prove the stronger theorem.

Theorem 5.7 (Baby Peter-Weyl Theorem)**.** *Let G be a compact Hasudorff group, and let $1 \neq x \in G$, then there exists a $\tau(G)$-invariant subspace on which x acts non-trivially.*

Proof. Assume for contradiction that $\tau(x) = \mathrm{id}$ on every $\tau(G)$-invariant subspace. In particular, by the claim and the spectral theorem, for every $g \in L^2(G)$, $\tau(x) = \mathrm{id}$ on the image of T_g.

Let us build a function g for which it fails. Let U be a compact identity neighborhood such that $x \notin U^2$. Let $g = \chi_U$ be supported in U, then $T_g(g) = g * g$ is support on U^2. A computation shows that $g * g(1) = \mu(U) > 0$ while $g * g(x) = 0$, and so $\tau(x)(g * g) \neq g * g$. ☐

Corollary 5.8 (Gleason-Yamabe for compact groups)**.** *Let G be a compact group, then for all identity neighborhood U there exists a normal $N \lhd G$ contained in U such that G/N is a compact linear group.*

Fact 5.9. Closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ are Lie.

Proof of Corollary 5.8. Assume without loss of generality that U is open. By Theorem 5.7, for every $x \notin U$ there exists a (continuous) finite dimensional representation $\rho_x : G \to \mathrm{GL}(V)$ such that $\rho_x(x) \neq \mathrm{id}$. By continuity, $\rho_x \neq \mathrm{id}$ on a neighborhood of x. By compactness of $G - U$, there are finitely many finite-dimensional representation $\{\rho_i : G \to \mathrm{GL}(V_i)\}_{i=1}^n$ such that every $x \in G - U$ is non-trivial in at least one of them. Therefore, $\rho = \bigoplus_{i=1}^n \rho_i : G \to \bigoplus_i \mathrm{GL}(V_i) \leq \mathrm{GL}(\bigoplus_{i=1}^n V_i)$ is a representation such that $\ker(\rho) \leq U$, and whose image is a compact subgroup of $\mathrm{GL}(\bigoplus_{i=1}^n V_i)$, as desired. ☐

We end this subsection with the statement of the full Peter-Weyl Theorem. For this, recall that a representation $G \to \mathrm{GL}(V)$ is *irreducible* if it has no non-trivial invariant subspaces.

Theorem 5.10 (Peter-Weyl Theorem)**.** *Let G be a compact Hausdorff group. Then $L^2(G) \simeq \bigoplus V^{\dim(V)}$ as representations, where the sum runs over all finite dimensional irreducible representations V of G up to isomorphism. More accurately, $\tau \simeq \bigoplus \rho^{\oplus \dim(\rho)}$.*

The example one should have in mind is $G = \mathbb{R}/\mathbb{Z}$. By Fourier series we know that $\{f_n(t) = e^{2\pi ikt}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(G)$, i.e

$$L^2(G) = \bigoplus_{k\in\mathbb{Z}} \mathbb{R} f_k.$$

Indeed, all irreducible representations of $\mathbb{R}/\mathbb{Z}$ are 1-dimensional (since it is abelian), and are given by the homomorphisms $f_k : G \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$.

Exercise 5.11 (Peter-Weyl for compact abelian groups, a.k.a Fourier analysis on compact abelian groups)**.** Let G be a compact abelian group. Denote by $\hat{G} = \mathrm{Hom}(G, \mathbb{S}^1)$ the space of continuous homomorphisms $\chi : G \to \mathbb{S}^1 \subset \mathbb{C}$ with the uniform convergence topology (i.e sup topology). Show that:

1. $C(G) = \overline{\mathrm{Span}_{\mathbb{C}}(\hat{G})}$ in the uniform convergence topology. (Hint: Stone-Weierstrass)
2. $L^2(G) = \bigoplus_{\chi\in\hat{G}} \mathbb{C}\chi$, i.e $\hat{G}$ forms an orthonormal basis for $L^2(G)$. (Hint: show that $\int \chi(g) d\mu(g) = 0$ for all $\chi \not\equiv 1$)

$\hat{G}$ is called the *Pontryagin dual* of G and elements of $\hat{G}$ are called *characters* of G.

5.3 From compact to non-compact abelian groups

Definition 5.12. A subgroup $H < G$ is *cocompact* if G/H is compact.

Exercise 5.13. For a locally compact group G, $H < G$ is cocompact if and only if there exists $K \subseteq G$ cocompact such that $G = HK$.

Lemma 5.14. *Let G be a locally compact group, then G has a subgroup which has a cocompact finitely generated subgroup.*

Proof. Let K be a symmetric compact 1 neighborhood. By compactness, there exists a finite set $F \subset G$, such that $K^2 \subseteq KF$. Without loss of generality, we can assume that $F \subset K^{-1}K^2 = K^3$. If we denote by G' the open subgroup generated by K and by H the subgroup of G' generated by F, then

$$K^n H = KH.$$

Therefore $G' \leq KH$. We deduce that $G' = KH$, as desired. □

In the abelian case we can improve the situation.

Definition 5.15. A *cocompact lattice* is a discrete cocompact subgroup.

Proposition 5.16. *Let G be a locally compact abelian group with a cocompact finitely generated subgroup, then G has a finitely generated lattice.*

For this we will need the following lemma.

Lemma 5.17. *Let G be a locally compact group, and let $g \in G$. Then, $\langle g \rangle$ is discrete or precompact.*

Proof. By replacing G with $\overline{\langle g \rangle}$ if necessary, we may assume that $\langle g \rangle$ is dense in G. Assume that it is not discrete, then there is a sequence $1 \neq g^{n_k} \to 1$. We may assume that $n_k \to \infty$ (by changing g^n to g^{-n} if needed).

Let U be a compact symmetric 1 neighborhood. Then U^3 can be covered by finitely many translates $x_i U$ $i = 1, \dots, m$. Since $\overline{\langle g \rangle g} = G$, we may find powers $r_1, \dots, r_m$ such that $g^{r_i} U^2$ cover U^3. Using the sequence g^{n_k} we can cover U^3 with $g^{s_i} U^2$ where $s_1, \dots, s_m > 0$. This means that if $g^n \in U^3$ then $g^{n - s_i} \in U^3$ for some $1 \leq i \leq m$. In other words, the set $\{n \in \mathbb{Z} | g^n \in U^3\}$ is left-syndetic (i.e has bounded gaps as $n \to -\infty$). Similarly, it is also right-syndetic, and hence syndetic (has bounded gaps). This means that there exists M such that $\langle g \rangle \in U \cup gU \cup \dots \cup g^M U$, which implies that it is precompact. □

Proof of Proposition 5.16. By induction on the 'rank' of G, i.e the minimal number of generators of a cocompact subgroup. If G has rank 0, then G is compact, and the trivial subgroup is a cocompact lattice.

Otherwise, G has rank r. I.e, it has a dense subgroup generated by $e_1, \dots, e_r \in G$. By Lemma 5.17, $\langle e_1 \rangle$ is discrete or precompact. If it is discrete, then $G / \langle e_1 \rangle$ is an abelian group that has rank $\leq r - 1$, and therefore has a finitely generated cocompact lattice H. Then $H \langle e_1 \rangle$ is a cocompact lattice in G. The same works if any of $\langle e_i \rangle$ is discrete. So, we may assume that all of $\langle e_1 \rangle, \dots, \langle e_r \rangle$ are precompact, but since G is abelian $\langle e_1, \dots, e_r \rangle = \langle e_1 \rangle \cdots \langle e_r \rangle$ is precompact. Thus G is compact (as it has a precompact cocompact subgroup), and again the trivial group is a cocompact lattice. □

Finally, let us deduce Gleason-Yamabe for locally compact abelian groups from Proposition 5.16, using the following fact.

Fact 5.18. A topological group which is "locally isomorphic to a Lie group" is a Lie group. (We will prove this later in the course)

Theorem 5.19 (Gleason-Yamabe for abelian groups)**.** *Let G be a locally compact abelian group, then for all identity neighborhood U there exists a compact normal subgroup N such that G/N is isomorphic to a Lie group.*

Proof. By the above discussion we can find $G' \leq G$ open with a discrete lattice $\Gamma \leq G'$. By shrinking U we may assume that it is compact, symmetric and $U^2 \cap \Gamma = 1$. Let $\pi : G' \to G'/\Gamma$ be the quotient map, note that π is injective on U. Since G'/Γ is a compact Hausdorff group, we can apply the Gleason-Yamabe Theorem for compact groups and find a subgroup $H \subset \pi(U)$ such that G'/U is a Lie group.

Exercise 5.20. $N = \pi^{-1}(U) \cap U$ is a compact subgroup of G.

Let $\phi : G' \to G'/N$, then $(G'/N)/\phi(\Gamma)$ is isomorphic to $(G'/\Gamma)/H$ and is thus a Lie group. This means that G'/N is locally a Lie group. It is an open subgroup of G/N which is therefore also locally a Lie group. By the fact above, it is a Lie group. □

6 Local Groups and the Exponential map

6.1 Local groups and Euclidean local groups

Let G be a topological group, and let U be an open identity neighborhood. Then the group operations define partial operations on U, which still satisfy the group axioms whenever all involved elements are defined. The obtained object is a 'local group' in the following sense.

Definition 6.1 (Local group)**.** A *local group* $G = (G, \Omega, \Lambda, 1, \cdot, ^{-1})$ is a topological space G with open sets $1 \in \Lambda$ and $\{1\} \times G \cup G \times \{1\} \subseteq \Omega \subseteq G \times G$, a partially defined multiplication $\cdot : \Omega \to G$ and a partially defined inverse $()^{-1} : \Lambda \to G$ which satisfies associativity, identity and inverse, whenever all the involved operations are defined (e.g, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ whenever both expressions are well-defined).

Remark 6.2.

1. Groups are local groups.
2. Open identity neighborhoods of local groups (with the restricted operations) are local groups.
3. A local morphism of local groups G, H is a map $\phi : U \to H$ from an identity neighborhood of G to H that respects the local group operations. We will say that two local groups are locally isomorphic if they have a local morphism with a local inverse.
4. Much of the course could have been done from the perspective of studying local group. E.g. locally compact local groups have Haar measure etc. In fact, every locally compact group is locally isomorphic to a locally compact group.

We will restrict our attention to *Euclidean local groups* – that is, local groups whose underlining space is an open subset of $\mathbb{R}^n$ and identity element 0 (but an arbitrary group law). In particular if G is a locally Euclidean group, then it has an open identity neighborhood U and a homeomorphism $\phi : U \to V$ to an open subset V of $\mathbb{R}^n$. Without loss of generality we may assume $\phi(1) = 0$ and define a partial multiplication $*$ on V by

$$x * y = \phi(\phi^{-1}(x) \cdot \phi^{-1}(y))$$

where $\cdot$ is the multiplication in G. This gives $V = (V, \Omega, \Lambda, 0, *, ^{-1})$ a structure of a Euclidean local group.

Our focus in the next chapters will be on improving the regularity of the multiplication – i.e. continuity / differentiability / smoothness / analyticity of the multiplication as a map from an open subset of $\mathbb{R}^{2n} \to \mathbb{R}^n$). Our starting point is the notion of $C^{1,1}$ *local groups* defined below. And our first goal is to show that the multiplication in a $C^{1,1}$ local groups can be made real-analytic, and in particular smooth, making it a *local Lie group* (also defined below). This is the essence of the Baker-Campbell-Hausdorff Formula (Theorem **??**).

Definition 6.3. A $C^{1,1}$ *local group* is a Euclidean local group $(V, \Omega, \Lambda, 0, *, ^{-1})$ where $V \subset \mathbb{R}^n$ is open, and if[3]

$$x * y = x + y + O(|x||y|)$$

for all sufficiently small $x, y \in V$ (the constant in O-notation may depend on V but is uniform in x, y).

Similarly, a *local Lie group* is a Euclidean local group in which the group operation is smooth (i.e C^∞).

6.2 The exponential map for matrices

Let $G = \mathrm{GL}(n, \mathbb{C})$. We can define $\exp : M(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

It is not too hard to show that exp is well-defined, real analytic (in fact, complex analytic), and satisfies $\exp((s+t)A) = \exp(sA)\exp(tA)$ for all $s, t \in \mathbb{C}$ and $A \in M(n, \mathbb{C})$. This shows that $t \mapsto \exp(tA)$ is a continuous group homomorphism $\mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$. Surprisingly, the converse holds true as well.

Proposition 6.4. *Let $\phi : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ be a continuous homomorphism. Then, there exists $A \in M(n, \mathbb{C})$ such that $\phi(t) = \exp(tA)$ for all $t \in \mathbb{R}$.*

Proof. It suffices to show that there exists A such that $\phi(t) = \exp(tA)$ for small enough t.

Since exp is real analytic (with derivative = id), it is onto a small enough neighborhood of I (which by changing the parametrization of ϕ, we may assume includes $\phi(1)$). Let A be such that $\phi(1) = \exp(A)$.

Since both ϕ, exp are homomorphisms we have

$$\phi(1/2)^2 = \phi(1) = \exp(A) = \exp(A/2)^2.$$

But $A \mapsto A^2$ is a diffeomorphism near 1 in $\mathrm{GL}(n, \mathbb{R})$. and so $\phi(1/2) = \exp(A/2)$. Repeating this we get $\phi(1/2^n) = \exp(A/2^n)$. And by the homomorphism property, we get $\phi(t) = \exp(tA)$ for all dyadic number $t \in \mathbb{R}[1/2]$. The dyadic numbers are dense in $\mathbb{R}$, and so by continuity of both ϕ, exp we get $\phi(t) = \exp(tA)$ for all $t \in \mathbb{R}$. □

In particular, it shows how every continuous homomorphism in $\mathrm{GL}(n, \mathbb{C})$ can be promoted to a real-analytic homomorphism.

3

Notation. Use $O(f(x))$ to denote some function which in norm/absolute value is $\leq K \cdot f(x)$ for some $K \in \mathbb{R}$.

6.3 Estimates for $C^{1,1}$ local groups

Let G be a $C^{1,1}$ local group. Our first goal is to make G into a *radially homogeneous* $C^{1,1}$ local group, where *radially homogeneous* stands for $sx * tx = (s+t)x$ for all $x \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ such that sx, tx are sufficiently small. For this we will define a map $\exp(x)$ that will 'straighten' our local group, and make it radially homogeneous.

We have seen how to define the map exp (on matrices) using a power series, but it will be more convenient for us to use the following limit

$$\exp(x) = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

or rather

$$\exp(x) = \lim_{n\to\infty} (1 + \frac{x}{2^n})^{2^n}.$$

We will start with some estimates.

Lemma 6.5. *Let V be a $C^{1,1}$ local group. Then, there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{R}^n$ such that $\sum_{i=1}^k |x_i|$ we have*

$$x_1 * \ldots * x_k = x_1 + \ldots + x_k + O\left(\sum_{1\leq<i<j\leq k} |x_i||x_j|\right) \tag{6.1}$$

where the constant in O can depend on ϵ but not on k.

Proof. We will prove it by induction on k. The base case, $k = 2$ is exactly the $C^{1,1}$ condition. But since will have to keep track of the constants in the O-notation, let us write it more concretely[4].

Let C, δ be such that for all $|x|, |y| \leq \delta$ we have

$$x * y = x + y + \angle(C|x||y|)$$

and in particular, $x * y = \angle(|x| + |y| + C|x||y|)$.

To illustrate the argument let us do the case $k = 3$:

$$\begin{aligned} x_1 * x_2 * x_3 &= x_1 * x_2 + x_3 + \angle(C|x_1 * x_2||x_3|) \\ &= x_1 + x_2 + x_3 + \angle(C|x_1||x_2|) + \angle(C(|x_1| + |x_2| + C|x_1||x_2|)|x_3|) \\ &= x_1 + x_2 + x_3 + \angle(C(|x_1||x_2| + |x_1||x_3| + |x_2||x_3|) + C^2|x_1||x_2||x_3|) \\ &= x_1 + x_2 + x_3 + \angle\left(C^{-1}\prod_{i=1}^{3}(1 + C|x_i|) - C^{-1} - \sum_{i=1}^{3}|x_i|\right) \end{aligned}$$

Let $\epsilon < \min\{\delta e^{-C}, \delta, 1\}$.

4

Notation (Unconventional)**.** We use $\angle(f(x))$ to denote some function which in norm/absolute value is $\leq f(x)$.

Induction Hypothesis: For all k and all $x_1, \dots, x_k$ such that $\sum_{i=1}^{k} |x_i| < \epsilon$ we have

$$x_1 * \dots * x_k = \sum_{i=1}^{k} x_i + \angle \left(C^{-1} \prod_{i=1}^{k} (1 + C|x_i|) - C^{-1} - \sum_{i=1}^{k} |x_i| \right) \tag{IH1}$$

$$= \sum_{i=1}^{k} x_i + \angle \left(C \exp(C) \sum_{1 \le i < j \le k} |x_i||x_j| \right) \tag{IH2}$$

and

$$|x_1 * \dots * x_k| \le C^{-1} \prod_{i=1}^{k} (1 + C|x_i|) - C^{-1} \tag{IH3}$$

$$\le \delta \tag{IH4}$$

Induction Step: Let $x_1, \dots, x_{k+1}$ such that $\sum_{i=1}^{k+1} |x_i| < \epsilon$. Then, by the Induction hypothesis (IH4) $|x_1 * \dots * x_k| < \delta$ and $|x_{k+1}| < \epsilon < \delta$. (IH1) (and consequently (IH3)) follows from the following:

$$\begin{aligned} x_1 * \dots * x_k * x_{k+1} &= x_1 * \dots * x_k + x_{k+1} + \angle \left(C |x_{k+1}| |x_1 * \dots * x_k| \right) \\ &= \sum_{i=1}^{k+1} x_i + \angle \left(C^{-1} \prod_{i=1}^{k} (1 + C|x_i|) - C^{-1} - \sum_{i=1}^{k} |x_i| \right) \\ &\quad + \angle \left(C |x_{k+1}| \left(C^{-1} \prod_{i=1}^{k} (1 + C|x_i|) - C^{-1} \right) \right) \\ &= \sum_{i=1}^{k+1} x_i + \angle \left(C^{-1} \prod_{i=1}^{k+1} (1 + C|x_i|) - C^{-1} - \sum_{i=1}^{k=1} |x_i| \right) \end{aligned}$$

where the first estimate follows from the $C^{1,1}$ condition, the second estimate follows (IH1) of the induction hypothesis.

To get (IH2). Let us estimate the obtained error

$$\begin{aligned} C^{-1} \prod_{i=1}^{k} (1 + C|x_i|) - C^{-1} - \sum_{i=1}^{k} |x_i| &\le C \sum_{1 \le i < j \le k} |x_i||x_j| \prod_{l=1}^{k} (1 + C|x_l|) \\ &\le C \sum_{1 \le i < j \le k} |x_i||x_j| \exp \left(C \sum_{l=1}^{k} |x_l| \right) \\ &\le C \exp(C) \sum_{1 \le i < j \le k} |x_i||x_j| \end{aligned}$$

where the second inequality follows from $1 + a \le \exp(a)$ and the last inequality follows from $\sum_{i=1}^{k} |x_i| < \epsilon < 1$.

Similarly, (IH4) follows from estimating the error in (IH3)

$$\begin{aligned} C^{-1} \prod_{i=1}^{k} (1 + C|x_i|) - C^{-1} &\le \sum_{i=1}^{k} |x_i| \prod_{j=1}^{k} (1 + C|x_i|) \\ &\le \epsilon \exp(C\epsilon) \le \epsilon \exp(C) \le \delta. \end{aligned}$$

This concludes the proof of the lemma. □

Similarly, one proves the following.

Lemma 6.6. *Let V be a $C^{1,1}$ local group. Then, there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$, $n \in \mathbb{Z}$ and $x, y, z, w, x_1, \ldots, x_k, y_1, \ldots, y_k \in V$ such that*

$$|x|, |y|, |nz|, |nw|, \sum_{i=1}^{k} |x_i|, \sum_{i=1}^{k} |y_k| < \epsilon$$

we have

$$x^{*-1} = -x + O(|x|^2) \tag{6.2}$$

$$x * y * x^{*-1} * y^{*-1} = O(|x||y|) \tag{6.3}$$

$$x * y * x^{*-1} = y + O(|x||y|) \tag{6.4}$$

$$y * x^{*-1}, x^{*-1} * y = O(|x - y|) \tag{6.5}$$

$$x_1 * \ldots * x_k = y_1 * \ldots * y_k + O\left(\sum_{i=1}^{k} |x_i - y_i|\right) \tag{6.6}$$

$$\frac{1}{2}|n||z - w| \le |z^{*n} - w^{*n}| \le 2|n||z - w| \tag{6.7}$$

$$(zw)^{*n} = z^{*n} w^{*n} + O(|n|^2|z||w|) \tag{6.8}$$

where all constants in O can depend on ϵ but are uniform in $k, n, x, \ldots$

Exercise 6.7. Prove Lemma 6.6

6.4 The exponential map

Definition 6.8. Define the map $\exp : V' \to V$ for $V' \subset V$ small enough, by

$$\exp(x) = \lim_{n \to \infty} \left(\frac{1}{2^n} x\right)^{*2^n}$$

Lemma 6.9. *The limit is well-defined.*

Proof. We will show that $x_n := \left(\frac{1}{2^n} x\right)^{*2^n}$ is Cauchy. Denote by $a = \frac{y}{2^{n+1}}$. By Ineq. (6.7) of Lemma 6.6

$$|x_{n+1} - x_n| = |a^{2^{n+1}} - (2a)^{2^n}| = |(a^2)^{2^n} - (2a)^{2^n}| \le 2 \cdot 2^n \cdot |a^2 - 2a|,$$

and

$$a^2 - 2a \le C|a|^2$$

Combining the two we get

$$|x_{n+1} - x_n| = 2 \cdot 2^n \cdot C \cdot \left|\frac{x}{2^{n+1}}\right|^2 = \frac{C|x|^2}{2^{n+1}}$$

The geometric series converges, and hence x_n is Cauchy. □

Lemma 6.10. *The map* exp *has the following properties:*

(i) For all sufficiently small $x \in \mathbb{R}^n$ *we have* $\exp(x) = x + O(|x|^2)$.

(ii) For all sufficiently small x, y *we have*

$$\exp(x+y) = \lim_{n\to\infty} (\exp(x/2^n) * \exp(y/2^n))^{2^n}.$$

(iii) For all sufficiently small x, y *we have*

$$\exp(x+y) = \exp(x) * \exp(y) + O(|x||y|).$$

(iv) For all sufficiently small $x, y \in \mathbb{R}^n$ *we have*

$$|(\exp(x) - \exp(y)) - (x-y)| \le \frac{1}{2}|x-y|.$$

(v) exp *is a local homeomorphism.*

(vi) For $x \in \mathbb{R}^n$ *and* $s, t \in \mathbb{R}$ *such that* $|sx|, |tx|$ *are sufficiently small, we have:*

$$\exp(sx + tx) = \exp(sx) * \exp(tx).$$

Proof. (i) follows from the proof of Lemma 6.9.

(ii) follows from

$$\left| \left(\frac{x}{2^n} * \frac{y}{2^n}\right)^{2^n} - \left(\frac{x+y}{2^n}\right)^{2^n} \right| \le 2 \cdot 2^n \cdot \left| \frac{x}{2^n} * \frac{y}{2^n} - \frac{x+y}{2^n} \right| \le O(2 \cdot 2^{-n} \cdot |x||y|)$$

where the first inequality is (6.7) of Lemma 6.6, and the second inequality is the $C^{1,1}$ condition.

(iii) follows from (ii) and (6.8) of Lemma 6.5

To prove (iv) first note that by (iii) we have,

$$\begin{aligned} &\exp(x) = \exp(x-y) + \exp(y) + O(|x-y||y|) \\ \implies &\exp(x) - \exp(y) = \exp(x-y) + O(|x-y||y|) \end{aligned}$$

From (i) we have,

$$\exp(x-y) = x - y + O(|x-y|^2)$$

Combining the two we get

$$\exp(x) - \exp(y) = x - y + O(|x-y|^2) + O(|x-y||y|)$$

and (iv) follows.

Clearly $\exp(x)$ is continuous. Let $x_0 \in \mathbb{R}^n$ small enough, by the Banach Fixed-Point Theorem for the contracting function $g_{x_0}(x) = x - \exp(x) + x_0$, there exists a unique $x \in \mathbb{R}^n$ (small enough) so that $g_{x_0}(x) = x$, that is $\exp(x) = x_0$. This shows that it is a local bijection, and (v) follows.

To prove (vi), it suffices to prove it for $s, t \in \mathbb{Q}$. That is, it suffices to prove that $\exp(kx) = \exp(x)^{*k}$ for all $k \in \mathbb{Z}$.

First let us prove it for $k = 2$,

$$\begin{aligned}\exp(2x) &= \lim(2x/2^n)^{2^n} = \lim(x/2^{n-1})^{2^n} = \lim(x/2^n)^{2^{n+1}} = \\ &= \lim(x/2^n)^{2^n} * (x/2^n)^{2^n} = \exp(x)^{*2}\end{aligned}$$

Now, fix some arbitrary k, observe first that $\exp(kx) = \exp(kx/2^n)^{*2^n}$ for all n by the case $k = 2$.

Hence,

$$\begin{aligned}&|\exp(kx) - (\exp(x/2^n)^{*2^n})^{*k}| = |\exp(kx/2^n)^{*2^n} - (\exp(x/2^n)^{*k})^{*2^n}| \\ &= 2 \cdot 2^n \cdot |\exp(kx/2^n) - \exp(x/2^n)^{*k}| \\ &\leq 2 \cdot 2^n \cdot \left(|\exp(kx/2^n) - (kx/2^n)| + |(kx/2^n) - \exp(x/2^n)^{*k}|\right) \\ &\leq 2 \cdot 2^n \cdot O\left(|kx/2^n|^2 + k^2|x/2^n|^2\right) \to 0\end{aligned}$$

where the last inequality follows from (i) and Lemma 6.5. □

Corollary 6.11. *Every $C^{1,1}$ local group has a neighborhood of the identity which is isomorphic (as a topological local group) to a radially homogeneous $C^{1,1}$ local group.*

7 The Adjoint representation and the Baker-Campbell-Hausdorff Formula

7.1 The adjoint representation

Following Corollary 6.11, we will assume throughout this section that our $C^{1,1}$ local group is radially homogeneous. We will prove that for such a local group the operation $*$ is real-analytic, and in particular, that it is a Lie local group. We will give a precise formula for the multiplication which is known as the Baker-Campbell-Hausdorff Formula.

Lemma 7.1. *For x, y, z small enough we have:*

$$x * y = x + y + O(|x + y||x|) \tag{7.1}$$
$$x * y = x + y + O(|x + y||y|) \tag{7.2}$$
$$x * y = x * z + O(|y - z|) \tag{7.3}$$
$$y * x = z * x + O(|y - z|) \tag{7.4}$$

Proof. The (7.1) follows from $C^{1,1}$ of $x = (x * y) * (-y)$. Others are done in a similar fashion. □

Lemma 7.2 (Adjoint representation)**.** *For all sufficiently small x the map $\mathrm{Ad}_x : \mathbb{R}^d \to \mathbb{R}^d$ defined by $\mathrm{Ad}_x(y) = x * y * (-x)$ is linear for sufficiently small y.*

Proof. Since Ad_x is continuous it suffices to show that it is additive, that is,

$$x * (y + z) * (-x) = (x * y * (-x)) * (x * z * (-x))$$

for sufficiently small x, y, z.

$$\begin{aligned}
y + z &= (y/n + z/n)^n \\
\implies x * (y + z) * (-x) &= x * (y/n + z/n)^n * (-x) \\
&= (x * (y/n + z/n) * (-x))^n \\
&= n(x * (y/n + z/n) * (-x)) \\
&= n(x * ((y/n) * (z/n) + O(1/n^2)) * (-x)) \\
\text{Lemma 7.1} &= n(x * (y/n) * (z/n) * (-x) + O(1/n^2)) \\
&= n(x * (y/n) * (-x) * x * (z/n) * (-x) + O(1/n^2)) \\
&= n(x * (y/n) * (-x) + x * (z/n) * (-x) + O(1/n^2) + O(1/n^2)) \\
&= x * y * x + x * z * (-x) + O(1/n)
\end{aligned}$$

Take $n \to \infty$. □

Observation 7.3.
- $\| \mathrm{Ad}_x - I \| = O(|x|)$
- $\| \mathrm{Ad}_x - \mathrm{Ad}_y \| = O(|x - y|)$

- $\mathrm{Ad}_{x*y} = \mathrm{Ad}_x * \mathrm{Ad}_y$, i.e, $x \mapsto \mathrm{Ad}_x$ is a local homomorphism.
- $t \mapsto \mathrm{Ad}_{tx}$ is a continuous homomorphism to $\mathrm{GL}(\mathbb{R}^d)$.
- Proposition 6.4 $\implies$ there exists a linear transformation $\mathrm{ad}_x : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathrm{Ad}_{tx} = \exp(t\,\mathrm{ad}_x)$, and in particular $\mathrm{ad}_x = \frac{d}{dt}\mathrm{Ad}_{tx}|_{t=0}$.
- $\implies \mathrm{ad}_{x+y} = \mathrm{ad}_x + \mathrm{ad}_y$ (since $\mathrm{Ad}_{tx} * \mathrm{Ad}_{ty} = \mathrm{Ad}_{t(x+y)} + O(|t|^2)$) and $\mathrm{ad}_{tx} = t\,\mathrm{ad}_x$.
- Hence ad_x is linear in x. I.e, $\mathrm{ad}_x(y) = [x,y]$ where $[,]$ is a bilinear form $\mathbb{R}^d \to \mathbb{R}^d$.

Corollary 7.4. *The map $x \to \mathrm{Ad}_x$ is real analytic.*

7.2 The Baker-Campbell-Hausdorff formula

Working under the assumption of a radially homogeneous $C^{1,1}$ group.

Lemma 7.5. *For x, y sufficiently small we have*

$$x * y = x + F(\mathrm{Ad}_x)y + O(|y|^2)$$

where

$$F(z) = \frac{z \log z}{z-1}.$$

Proof. Let $z = x * y - x$. Note that $z = O(|y|)$ and $y = (-x) * (x+z)$. It suffices to show that

$$x * (x+z) = \frac{1-\exp(-\mathrm{ad}_x)}{\mathrm{ad}_x} z + O(|z|^2) \tag{7.5}$$

Assuming we have shown the above equality, since

$$\frac{1-\exp(-\mathrm{ad}_x)}{\mathrm{ad}_x} = \frac{\mathrm{Ad}_x - 1}{\mathrm{Ad}_x \log(\mathrm{Ad}_x)} = F(\mathrm{Ad}_x)^{-1}$$

Rewriting (7.5) we get

$$y = F(\mathrm{Ad}_x)^{-1}(x * y - x) + O(|y|^2)$$

and the claim follows by inverting $F(\mathrm{Ad}_x)$.

to prove (7.5), let n be a large integer, expand the LHS as a telescopic sum

$$\sum_{j=0}^{n-1} (-\frac{j+1}{n}x) * (\frac{j+1}{n}x + \frac{j+1}{n}z) - (-\frac{j}{n}x) * (\frac{j}{n}x + \frac{j}{n}z)$$

The first summand is

$$(-\frac{j}{n}x) * (-\frac{1}{n}x) * (\frac{1}{n}x + \frac{1}{n}z) * (\frac{j}{n}x + \frac{j}{n}z).$$

From Lemma 7.1 $(-\frac{1}{n}x) * (\frac{1}{n}x + \frac{1}{n}z) = \frac{1}{n}z + O(\frac{|z|}{n^2})$ and thus the preceding expression becomes

$$(-\frac{j}{n}x) * (\frac{1}{n}z) * (\frac{j}{n}x + \frac{j}{n}z) + O(\frac{|z|}{n^2}).$$

By definition of Ad it equals

$$(\mathrm{Ad}_{-\frac{j}{n}x}\frac{1}{n}z) * (-\frac{j}{n}x) * (\frac{j}{n}x + \frac{j}{n}z) + O(\frac{|z|}{n^2}).$$

By Lemma 7.1 $(-\frac{j}{n}x) * (\frac{j}{n}x + \frac{j}{n}z) = O(|z|)$ and by the $C^{1,1}$ the preceding expression becomes

$$(\mathrm{Ad}_{-\frac{j}{n}x}\frac{1}{n}z) + (-\frac{j}{n}x) * (\frac{j}{n}x + \frac{j}{n}z) + O(\frac{|z|^2}{n}) + O(\frac{|z|}{n^2}).$$

Inserting this in the telescopic sum we get that the LHS of 7.5 is

$$\sum_{j=0}^{n-1} \mathrm{Ad}_{-\frac{j}{n}x}\frac{1}{n}z + O(|z|^2) + O(\frac{|z|}{n})$$

Writing $\mathrm{Ad}_{-\frac{j}{n}x} = \exp(-\frac{j}{n}\,\mathrm{ad}_x)$ and letting $n \to \infty$, the Riemann sum converges to the Riemann integral

$$\begin{aligned}(-x) * (x+z) &= \int_0^1 \exp(-t\,\mathrm{ad}_x)z\,dt + O(|z|^2)\\ &= \frac{-\exp(-t\,\mathrm{ad}_x)z}{\mathrm{ad}_x}\Big|_{t=0}^{t=1}\\ &= \frac{1-\exp(-\mathrm{ad}_x)}{\mathrm{ad}_x}z + O(|z|^2)\end{aligned}$$

as desired. □

As a corollary we get the following formula for multiplication.

Theorem 7.6 (Baker-Campbell-Hausdorff Formula). *Let V be a radially homogeneous $C^{1,1}$ local group, then for x, y sufficiently small, one has*

$$x * y = x + \int_0^1 F(\mathrm{Ad}_x\,\mathrm{Ad}_{ty})y\,dt. \tag{7.6}$$

As a consequence, the multiplication in a radially homogeneous $C^{1,1}$ local group is real-analytic.

Proof. Use the telescopic sum

$$x * y = x + \sum_{j=0}^{n-1} x * (\frac{j+1}{n}y) - x * (\frac{j}{n}y)$$

From Lemma 7.5 we get that the left summand is

$$\begin{aligned} x * (\frac{j+1}{n} y) &= x * (\frac{j}{n} y) * (\frac{1}{n} y) \\ &= x * (\frac{j}{n} y) + F(\mathrm{Ad}_x \, \mathrm{Ad}_{\frac{j}{n} y})(\frac{1}{n} y) + O(1/n^2) \end{aligned}$$

and conclude that

$$x * y = x + \sum_{j=0}^{n-1} F(\mathrm{Ad}_x \, \mathrm{Ad}_{\frac{j}{n} y})(\frac{1}{n} y) + O(1/n).$$

As $n \to \infty$ these Riemann sums converge to the desired integral.

Since the RHS of the Baker-Campbell-Hausdorff Formula is real analytic, we get the consequence. □

8 Local and global Lie groups

8.1 Lie local groups

Theorem 8.1 (Lie's First Theorem)**.** *For a Lie local group,* exp *is a local diffeomorphism.*

Proof. Since ϕ :"$\mathbb{R} \to \mathbb{R}^d$" defined by $\phi(t) = \exp(tx)$ is a homomorphism with

$$\frac{d}{dt}|_{t=0}\phi = x$$

(since $\exp(x) = x + O(|x|^2)$), we have that ϕ satisfies the ODE

$$\frac{d}{dt}\phi(t) = DL_{\phi(t)}(x)$$

where L is the left multiplication map. That is, $f(t,x) = \exp(tx)$ is a solution to the ODE

$$\frac{d}{dt}f(t,x) = F(f(t,x),x)$$

for some smooth F. By the existence and uniqueness theorem for solutions to ODE, we know that the (unique) solution $f(t,x) = \exp(tx)$ is differentiable k times around $(0,0)$, for all k. By using the homomorphism property of $\exp(tx)$ we get that it is smooth in a neighborhood of $(0,0)$, and also at neighborhood of $(1,0)$. Hence exp is smooth.

The map $D_0 \exp(x) = x$ hence, by the inverse function theorem it is a local diffeomorphism. □

In a similar way to Proposition 6.4, one proves the following.

Proposition 8.2. *Let V be a Lie local group, and let $\phi : \mathbb{R} \to V$ be a local continuous homomorphism. Then there exists a unique $x \in V$ such that $\phi(t) = \exp(tx)$ for small enough t.*

Exercise 8.3. Prove Proposition 8.2. (Hint: follow the same proof as Proposition 6.4.

Let us use this proposition to show how to upgrade continuous homomorphisms to smooth homomorphisms.

Proposition 8.4. *Let G, H be local Lie groups, and let $\Phi : G \to H$ be a continuous homomorphism. Then Φ is smooth.*

Proof. Let $x \in G$. The map $t \mapsto \Phi(\exp(tx))$ is a continuous homomorphism, and hence by the previous proposition there exists a unique $L(x) \in H$ such that $\Phi(\exp(tx)) = \exp(tL(x))$. Since exp are local diffeomorphisms, it suffices to show that the map L is smooth (in a 0 neighborhood). In fact, L is linear.

For all $s \in \mathbb{R}$ small enough, we clearly have $L(sx) = sL(x)$. For additivity, let $x, y \in G$,

$$\Phi(\exp(t(x+y))) = \Phi(\lim_{n\to\infty}(\exp(tx/2^n)\exp(ty/2^n))^{2^n} \tag{8.1}$$
$$= \lim_{n\to\infty}(\Phi(\exp(tx/2^n))\Phi(\exp(ty/2^n)))^{2^n} \tag{8.2}$$
$$= \lim_{n\to\infty}(\exp(tL(x)/2^n)\exp(tL(y)/2^n))^{2^n} \tag{8.3}$$
$$= \exp(t(L(x)+L(y))). \tag{8.4}$$

□

Corollary 8.5. *A local group can have at most one smooth structure.*

8.2 Smooth manifolds and Lie groups

Definition 8.6. A topological space M is an *n-topological manifold* if it is Hausdorff and locally homeomorphic to $\mathbb{R}^2$.

A *smooth atlas* on a topological manifold is an open cover $\{U_\alpha\}$ of M and homeomorphisms $\{\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n\}$ such that the transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth wherever they are defined. A topological manifold with a smooth atlas is a *smooth manifold*.

Example 8.7. 1. $\mathbb{R}^n$ with the obvious smooth structure.

2. Any open subset of a smooth manifold is a smooth manifold. In particular $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold.

3. If M and N are smooth manifolds (of dimension k and l respectively) then $M \times N$ is a smooth manifold (of dimension $n = k + l$) with the atlas $\phi_\alpha \times \psi_\beta \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$.

A *smooth map* between smooth manifolds is a map $f : M \to N$ (with smooth atlases $\{\phi_\alpha\}_\alpha, \{\psi_\beta\}_\beta$ repsectively) such that $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth whenever it is defined. We will denote by $C^\infty(M, N)$ the set of all smooth maps $M \to N$, and by $C^\infty(M) = C^\infty(M, \mathbb{R})$.

A smooth map $f : M \to N$ is a *diffeomorphism* of smooth manifolds if it has a smooth inverse.

Definition 8.8. A *Lie group* is a topological group G which is a smooth manifold, and such that the group operation $\cdot : G \times G \to G$ and inverse map $\cdot^{-1} : G \to G$ are smooth maps.

Example 8.9. $\mathbb{R}^n$, $GL(n, \mathbb{R})$ are Lie group.

Fact 8.10. All closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ are smooth manifolds. In particular, the subgroups $\mathrm{SL}(n, \mathbb{R}), \mathrm{O}(n, \mathbb{R}), \mathrm{U}(n, \mathbb{R}), \mathrm{SU}(n, \mathbb{R}), \ldots$ are Lie groups.

Every Lie group is locally (diffeomorphically) isomorphic to a Lie local group, in particular, it can be made into a radially homogeneous $C^{1,1}$ local group using the map exp. We have seen that exp is a local diffeomorphism. Therefore, every Lie group is locally (diffeomorphically) isomorphic to a radially homogeneous Lie local group. In fact, we know that it is then automatically real-analytic.

Theorem 8.11. *Let G be a topological group, if G is locally isomorphic to a Lie local group, then G is a Lie group.*

Proof. Let $\phi' : U' \to V'$ be the local isomorphism identifying a 1-neighborhood U' of G with a Lie local group V'. Let us $\phi : U \to V$ be the restriction of ϕ' to a smaller 1-neighborhood $U \subseteq U'$ to be determined in the argument.

We define $\phi_g : gU \to V$ by $\phi_g = \phi \circ L_{g^{-1}}$. And let $\mathcal{F} = \{\phi_g\}_{g\in G}$ be an atlas on G.

The atlas $\mathcal{F}$ is smooth. Let gU, hU be two intersecting neighborhood, then $h^{-1}g \in U^2 \subseteq U'$ then the transition map

$$\phi_h \circ \phi_g^{-1} = \phi \circ L_{h^{-1}} \circ L_g \circ \phi^{-1} = \phi \circ L_{h^{-1}g} \circ \phi^{-1}$$

If we denote by $y = \phi'(h^{-1}g)$ then the transition map becomes

$$\phi_h \circ \phi_g^{-1} = L_y$$

which is a diffeomorphism by assumption on V.

To show that G with the smooth structure $\mathcal{F}$ is a Lie group, we have to show that the multiplication map $m : G \times G \to G$ and inverse map $i : G \to G$ are smooth.

The inverse map is smooth. To show that i is smooth it suffices to show that $\phi_g \circ i \circ \phi_{g^{-1}}^{-1}$ is a smooth map around 0.

$$\begin{aligned} \phi_g \circ i \circ \phi_{g^{-1}}^{-1} &= \phi \circ L_{g^{-1}} \circ i \circ L_{g^{-1}} \circ \phi^{-1} && (8.5)\\ &= \phi \circ c_{g^{-1}} \circ i \circ \phi^{-1} && (8.6)\\ &= (\phi \circ c_{g^{-1}} \circ \phi^{-1}) \circ (\phi \circ i \circ \phi^{-1}) && (8.7) \end{aligned}$$

The map $\phi \circ i \circ \phi^{-1}$ is simply the inverse map in V, hence it is smooth. The map $\phi \circ c_{g^{-1}} \circ \phi^{-1}$ is a local continuous homomorphism of V, which by Proposition 8.4 is smooth.

Exercise 8.12. Show that the multiplication map is smooth.

□

Exercise 8.13. Show that any continuous homomorphism between Lie groups is smooth. Deduce that there is a unique smooth structure on a topological group that makes it into a Lie group.

9 Topological Vector Spaces

Definition 9.1. A *topological vector space* (over $\mathbb{R}$)is a vector space V over $\mathbb{R}$ with a topology such that $+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$ are continuous.

Our goal in this section is to prove the following theorem.

Theorem 9.2. *Let V be a locally compact Hausdorff topological vector space. Then V is isomorphic (as a topological vector space) to $\mathbb{R}^d$ for some finite d.*

Lemma 9.3. *Every finite dimensional Hausdorff topological space has the usual topology.*

Proof. Let V be a finite dimensional Hausdorff topological space. Let $e_1, \dots, e_d$ be a basis for V. Consider the bijective map $T : \mathbb{R}^d \to V$ given by $T(\alpha_1, \dots, \alpha_d) = \alpha_1 e_1 + \dots + \alpha_d e_d$. Since V is a topological space, T is continuous. It suffices to show that T is open. It suffices to show that V has some open neighborhood U of 0 such that $T^{-1}(U)$ is bounded (since by dilating and translating it will give us the a basis).

Let F be the unit sphere in $\mathbb{R}^d$, F is compact. Hence, $T(F)$ is closed, and we can find a 0 neighborhood U' such that $U' \cap T(F) = \varnothing$. By continuity of scalar multiplication, we may find an open 0 neighborhood U and $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cdot U \subset U'$. It follows that $T^{-1}(U) \subset B(0, 1/\epsilon) \subset \mathbb{R}^d$ (since $(-\epsilon, \epsilon) \cdot T^{-1}(U)$ must avoid the unit sphere). □

Corollary 9.4. *In a Hausdorff topological vector space, every finite dimensional subspace is closed.*

Proof. Let $W \subset V$ be a finite dimensional subspace of V. Let $v \in V - W$. Then $\mathbb{R}v + W$ is a finite dimensional topological space, which is therefore homeomorphic to $\mathbb{R}^d$. It follows that v has a neighborhood which does not intersect W. □

Proof of Theorem 9.2. Let K be a compact 0 neighborhood of V. Clearly V is spanned by K.

Since $\frac{1}{2}K$ is also a 0 neighborhood, it follows by compactness that $K \subseteq S + \frac{1}{2}K$ for some finite S in K.

Let W be the subspace generated by S. So $K \subseteq W + \frac{1}{2}K$. Iterating we get that $K \subseteq W + \frac{1}{2^n}K$ for all n. Since for every open 0 neighborhood contains $\frac{1}{2^n}K$ for some large n, it follows that $K \subseteq \overline{W} = W$. Hence $W = V$. □

Exercise 9.5. Can you prove Theorem 9.2 using Gleason-Yamabe for abelian groups?

10 Gleason Metrics

Our goal is to show a sufficient condition for being radially homogeneous $C^{1,1}$ local group using metrics on groups.

Definition 10.1 (Gleason metric)**.** A *weak Gleason metric* d on a topological group G, is a left-invariant metric which generates the topology on G for which there exists some $C > 0$ such that:

(GM1) (Escape property) If $g \in G$ and $n \geq 1$ is such that $n\|g\| \leq C$ then $\|g^n\| \geq \frac{1}{c} n\|g\|$.

where $\|g\| = d(g, 1)$.

A *Gleason metric* d on a topological group G is a weak Gleason metric that satisfies in addition

(GM2) (Commutator estimate) If $g, h \in G$ are such that $\|g\|, \|h\| \leq \frac{1}{C}$ then $\|[g, h]\| \leq C\|g\|\|h\|$.

where $[g, h] = g^{-1}h^{-1}gh$.

Our goal is the following theorem.

Theorem 10.2. *Let G be a locally compact topological group with a weak Gleason metric. Then, G is isomorphic to a Lie group.*

For most of this section we will assume G has a Gleason metric, and in the last subsection we will prove that a weak Gleason metric can be upgraded to a Gleason metric using convolution.

10.1 Estimates on Gleason metrics

In what follows, let G be a locally compact group with a Gleason metric d. In the following section we will need the following estimates for a Gleason metric.

Lemma 10.3.

1. *(approximate right invariance) If g, h, k are small enough, then $d(gk, hk) \sim d(g, h)$.*
2. *(commutation estimate) If g, h are small enough, then $d(gh, hg) = O(\|g\|\|h\|)$.*
3. *(power estimates) If $n \geq 1$ and $\|g\|, \|h\| \leq \epsilon/n$, then*

$$d(g^n h^n, (gh)^n) = O(n^2\|g\|\|h\|)$$

and

$$d(g^n, h^n) \sim nd(g, h)$$

Proof. From left invariance we get $\|g\| = \|g^{-1}\|$, and from the commutator estimate and left invariance we get $\|h^{-1}gh\| \sim \|g\|$. Hence the approximate right invariance follows by

$$d(gk, hk) = \|k^{-1}(h^{-1}g)k\| \sim \|h^{-1}g\| = d(g, h).$$

The commutation estimate follows directly from left invariance and the commutator estimate.

For the first estimate, It suffices to show that

$$d((gh)^i g^{n-i} h^{n-i}, (gh)^{i+1} g^{n-i-1} h^{n-i-1}) = O(n\|g\|\|h\|)$$

uniformly in $1 \le i < n$. By left invariance, and approximate right invariance we need to show

$$d(g^{n-i-1}h, hg^{n-i-1}) = O(n\|g\|\|h\|).$$

By the commutation estimate $d(g^{n-i-1}h, hg^{n-i-1}) \ll \|g^{n-i-1}\|\|h\| \ll n\|g\|\|h\|$ as desired.

Now, define $k = h^{-1}g$ then $\|k\| = d(g,h) \le 2\epsilon/n$. We have

$$d(h^n k^n, h^n) = \|k^n\| \sim n\|k\|$$

by the escape property. On the other hand,

$$d(g^n, h^n k^n) \ll n^2 \|h\|\|k\|$$

and the second power estimate follows. □

10.2 The space of 1-parameter subgroups

Let G be a topological group. A *1-parameter subgroup* is a continuous homomorphism $\phi : \mathbb{R} \to G$. Let us denote by $L(G)$ the set of 1-parameter subgroup in G with the compact-open topology. Recall that the compact-open topology is the topology generated by the sub-basis $V_{K,U} = \{\phi | \phi(K) \subset U\}$ for all compact $K \subset \mathbb{R}$ and all open $U \subseteq G$. If G is metrizable it is convenient to view this topology as the sup-metric on compact sets.

We will show that if G has a Gleason metric then the space $L(G)$ is in fact a locally compact topological vector space over $\mathbb{R}$, and as such must be isomorphic (as a topological vector space) to $\mathbb{R}^d$ for some d, and that the map $L(G) \ni \phi \mapsto \phi(1) \in G$ is a local isomorphism which makes G locally into a radially homogeneous $C^{1,1}$ local group, and thus into a Lie group.

Lemma 10.4. *There exists $\epsilon > 0$ small enough and $C > 0$ large enough such that for all $g \in G$ and $n \in \mathbb{N}$ such that $\|g^i\| \le \epsilon$ for $i = 1, \dots, n$, we have $\|g\| \le \frac{1}{n}$.*

Proof. Let $\delta = \min\{\frac{1}{2C}, 1\}$, and let $\epsilon = \frac{\delta}{C}$. Let $k = \max\{j : j\|g\| \le \delta\}$, it suffices to show that $k \ge n$.

For $k+1$ we have

$$\delta < (k+1)\|g\| \le 2k\|g\| \le 2\delta \le \frac{1}{C}.$$

Thus, by the escape property,

$$\|g^{k+1}\| \ge \frac{1}{C}(k+1)\|g\| > \frac{\delta}{C} = \epsilon$$

Therefore, if $\|g^i\| \le \epsilon$ for all $i = 1, \dots, n$ then $k \ge n$. □

Lemma 10.5. *$L(G)$ is locally compact.*

Proof. Let $\phi_0 \in L(G)$, We will use the Arzela-Ascoli theorem to prove that a the neighborhood

$$F := \{\phi \in L(G) | \sup_{t \in [-T,T]} d(\phi(t), \phi_0(t)) \leq \epsilon\}$$

of ϕ_0 is compact for small enough ϵ, T. Let us choose T so that $\|\phi_0(t)\| \leq \epsilon$ for all $t \in [-T, T]$.

Theorem 10.6 (Arzela-Ascoli)**.** *Let X be a compact Hausdorff space and Y a metric space. Then $F \subseteq C(X, Y)$ is compact in the compact-open topology if and only if it is equicontinuous, pointwise precompact and closed.*

Here $X = [-T, T] \subseteq \mathbb{R}$, $Y = (G, d)$. Clearly, F is closed, and since G is locally compact F is relatively compact for small enough ϵ. So it remains to show that F is equicontinuous. In fact, the functions in F are K-Lipschitz continuous for some K on $[-T, T]$, and thus equicontinuous.

Note that for all $\phi \in F$, we have $\|\phi(t)\| \leq 2\epsilon$ for all $t \in [-T, T]$. For small enough ϵ, Lemma 10.4 implies that $\|\phi(T/n)\| \leq 1/n$, thus $\|\phi(r)\| \leq |r|/T$ for all $r \in \mathbb{Q}$ and hence also for all $r \in \mathbb{R}$. By the homomorphism property $\|\phi(t) - \phi(s)\| = \|\phi(t-s)\| \leq |t-s|/T$ for all $t, s \in [-T, T]$, as desired. □

This proof also shows that all 1-parameter subgroups are locally Lipschitz.

Now let us endow $L(G)$ with a (real) vector space structure.

Scalar multiplication: Let $c \in \mathbb{R}$, and $\phi \in L(G)$. Then, define

$$c\phi(t) := \phi(ct).$$

Addition: For $\phi, \psi \in L(G)$, let us define $\phi + \psi$ using the limit

$$(\phi + \psi)(t) := \lim_{n \to \infty} (\phi(t/n)\psi(t/n))^n.$$

We still have to show that this limit exists and is in $L(G)$.

Lemma 10.7. *If $\phi, \psi \in L(G)$, then $\phi + \psi$ is well-defined and in $L(G)$.*

Proof. To show that the limit converges we will show that it is Cauchy (recall that G is complete since it is locally compact). We will prove the stronger claim

$$\sup_{m \geq 1} \sup_{1 \leq n' \leq n} d\Big((\phi(t/n)\psi(t/n))^{n'}, (\phi(t/nm)\psi(t/nm))^{n'm}\Big) \xrightarrow[n \to \infty]{} 0$$

Note that if the claim is true for $t/2$ then it is true for t (WHY?). Therefore, we may assume that t is small enough. Since ϕ, ψ are locally Lipschitz, we have $\|\phi(t/n)\|, \|\psi(t/n)\| \ll \epsilon/n$ for all n. Therefore, from Lemma 10.3 we have

$$d(\phi(t/n)\psi(t/n), (\phi(t/nm)\psi(t/nm))^m) \ll m^2(\epsilon/nm)^2 = \epsilon^2/n^2.$$

And again by Lemma 10.3

$$d((\phi(t/n)\psi(t/n))^{n'}, (\phi(t/nm)\psi(t/nm))^{n'm}) \ll n'\epsilon^2/n^2 \ll \epsilon/n \to 0$$

The above argument is uniform in t as long as t is small enough. Hence, it converges uniformly on compact to the limit. Hence, the limit is continuous. To prove that $\phi + \psi$ is a homomorphism, by the density of rationals, it suffices to show that

$$(\phi + \psi)(at)(\phi + \psi)(bt) = (\phi + \psi)((a + b)t)$$

and

$$(\phi + \psi)(-t) = (\phi + \psi)(t)^{-1}$$

for all $a, b \in \mathbb{N}, t \in \mathbb{R}$. The first follows from the computation

$$(\phi + \psi)(at) = \lim_{n\to\infty} (\phi(at/n)\psi(at/n))^n = \lim(\phi(t/n)\psi(t/n))^{an}$$

and the same for the other terms in the equation. Similarly,

$$\begin{aligned}(\phi + \psi)(-t) &= \lim_{n\to\infty} (\phi(-t/n)\psi(-t/n))^n \\ &= \lim_{n\to\infty} (\psi(t/n)\phi(t/n))^{-n} \\ &= \lim_{n\to\infty} \psi(t/n) \cdot (\phi(t/n)\psi(t/n))^{-n} \cdot \psi(t/n)^{-1} \\ &= (\phi + \psi)(t)^{-1}\end{aligned}$$

□

Lemma 10.8. $L(G)$ *is a topological vector space.*

Proof. Let us first show that $L(G)$ is a vector space over $\mathbb{R}$:

Associativity of addition. That is, we have to show that for all $t \in \mathbb{R}$ and $\phi, \psi, \eta \in L(G)$

$$((\phi + \psi) + \eta)(t) = (\phi + (\psi + \eta))(t).$$

But in fact, it suffices to prove it for small t.

We have seen that

$$d((\phi + \psi)(t), (\phi(t/n)\psi(t/n))^n) \ll \epsilon^2/n,$$

and

$$d((\phi + \psi)(t/n), \phi(t/n)\psi(t/n)) \ll \epsilon^2/n^2.$$

Similarly,

$$d(((\phi + \psi) + \eta)(t), ((\phi + \psi)(t/n)\eta(t/n))^n) \ll \epsilon^2/n$$

hence (by Lemma 10.3)

$$d(((\phi + \psi) + \eta)(t), (\phi(t/n)\psi(t/n)\eta(t/n))^n) \ll \epsilon^2/n.$$

A similar computation shows that

$$d((\phi + (\psi + \eta))(t), (\phi(t/n)\psi(t/n)\eta(t/n))^n) \ll \epsilon^2/n,$$

and the associativity follows.

Exercise 10.9. Prove the that $L(G)$ satisfies the remaining vector space axioms.

To show that it is a topological vector space. It is easy to see that the scalar multiplication is continuous. To show that addition is continuous it suffices to check that it is continuous at $(0,0)$. That is, for all $\epsilon > 0$ if ϕ, ψ satisfy $\sup_{t\in[-1,1]} \|\phi(t)\| \le \delta$, $\sup_{t\in[-1,1]} \|\psi(t)\| \le \delta$ then $\sup_{t\in[-1,1]} \|(\phi+\psi)(t)\| \le \epsilon$. Indeed, by Lipschitz continuity, $\|\phi(t)\| \ll \delta|t|$ and $\|\psi(t)\| \ll \delta|t|$ and therefore, $\|(\phi+\psi)(t)\| \ll \delta$. □

Now we can define an exponential map $L(G) \to G$ by $\exp(\phi) := \phi(1)$. Note that this map is continuous.

Proposition 10.10. *The image of any neighborhood of the origin in $L(G)$ is a neighborhood of the identity in G.*

Proof. Idea: to show that $L(G)$ is not trivial if G is non-discrete. Let $g_n \to 1$ in G. If we denote by $N_n = \lfloor \epsilon/\|g_n\| \rfloor$, then by the escape property $\|g^{N_n}\| \sim \epsilon$. Consider $\phi_n(t) := g_n^{\lfloor tN_n \rfloor}$ for a sequence $g_n \to 1$. The maps ϕ_n satisfy $\|\phi(t)\| \ll \epsilon|t| + \frac{\epsilon}{N_n}$ and the approximate homomorphism property

$$d(\phi_n(t+s), \phi_n(t)\phi_n(s)) \to 0.$$

ϕ_n are asymptotically equicontinuous, and by a generalization of Arzela-Ascoli they converge to a continuous homomorphism ϕ, with $\|\phi(1)\| \sim \epsilon$.

Exercise 10.11. Complete the details of this proof idea.

1. Prove the needed version of Arzela-Ascoli.
2. Show that ϕ_n are indeed 'asymptotically equicontinuous' and that a converging subsequence will converge to a homomorphism.

Let us proceed to the proof of the proposition. We may assume that the origin neighborhood K is compact and convex, and that $\exp(K)$ is contained in the ball of radius ϵ around the identity.

Assume that K is not an identity neighborhood, then there exists a sequence g_n of elements in $G - \exp(K)$ such that $g_n \to 1$. By compactness of $\exp(K)$ we can write $g_n = k_n h_n$ where $k_n \in \exp(K)$ are closest to g_n. That is, $\|h_n\| = d(g_n, k_n) = d(g_n, \exp(K)) \le \|g_n\|$. Hence $h_n, k_n \to 1$ as well.

Consider the sequence $\phi_n(t) := h_n^{\lfloor tN_n \rfloor}$ for $N_n = \lfloor \epsilon/\|k_n\| \rfloor$. As before, by passing to a subsequence we may assume that $\phi_n \to \phi \in L(G)$.

Let $\psi_n \in K$ be such that $\exp(\psi_n) = \psi_n(1) = k_n$, we have $\psi_n \to 0 \in L(G)$ (by the escape property $\|\psi_n(t)\| \ll t\|h_n\|$).

Now, consider the element $\exp(\psi_n + \frac{1}{N_n}\phi)$. Let us compute its distance from g_n.

By the power estimates of Lemma 10.3,

$$d(\exp(\psi_n + 1/N_n\phi), \exp(\psi_n)\exp(1/N_n\phi)) \ll \|k_n\|/N_n$$

By approximate right invariance Lemma 10.3 and triangle inequality,

$$d(\exp(\psi_n + 1/N_n\phi), g_n) \ll \|k_n\|/N_n + d(h_n, \exp(1/N_n\phi))$$

But from the power estimates of Lemma 10.3,

$$d(h_n, \exp(1/N_n\phi)) \ll d(h_n^{N_n}, \exp(\phi))/N_n = o(1/N_n)$$

and thus

$$d(\exp(\psi_n + 1/N_n\phi, g_n) = o(1/N_n)$$

For large enough n we may assume that $\psi_n + 1/N_n\phi \in K$ (recall that $\psi_n \to 0$ and K is an identity neighborhood), and so the distance from g_n to K is $o(1/N_n) = o(d(g_n, k_n))$. But this contradicts the minimality of k_n. □

Exercise 10.12. Prove that the exponential map $\exp : L(G) \to G$ a local homeomorphism.

Proposition 10.13. *G is locally a local $C^{1,1}$ group.*

Proof. Identify a local neighborhood of G with its pre-image in $L(G)$. By Theorem 9.2, $L(G)$ is homeomorphic to $\mathbb{R}^d$, with the norm

$$\|\phi\| := \lim_{n\to\infty} n\|\phi(1/n)\|.$$

Exercise 10.14. Show that this norm exists and generates the topology on $L(G)$, and that exp is locally a bi-Lipschitz map.

The $C^{1,1}$ inequality will follow from the power estimate of Lemma 10.3 as follows

$$d(\phi(1)\psi(1), (\phi(1/n)\psi(1/n))^n) \ll \|\phi(1)\|\|\psi(1)\|$$

for all n. Hence, by the definition of $\phi + \psi$,

$$d(\phi(1)\psi(1), (\phi + \psi)(1)) \ll \|\phi(1)\|\|\psi(1)\|$$

as desired.

10.3 From weak Gleason metrics to Gleason metrics

Finally let us show that a weak Gleason metric suffices.

Proposition 10.15. *Let G be a locally compact topological group, every weak Gleason metric on G is a Gleason metric.*

Proof. Idea: compare the weak Gleason metric to a left-invariant metric constructed using a bump function which comes from a convolution of a Lipschitz function.

Let $\epsilon > 0$ be small enough, to be chosen later. and let $\psi \in C_c(G)$ be the function supported on $B(1,\epsilon)$ defined by

$$\psi(x) = \max\left\{1 - \frac{\|x\|}{\epsilon}, 0\right\}.$$

And let

$$\phi(x) = \psi * \psi(x) = \int \psi(y)\psi(y^{-1}x)d\mu(y).$$

Let us denote by $\tau(g)$ the action of g on $C_c(G)$ by left multiplication, and denote by $\partial_g = 1 - \tau(g)$. The function ϕ gives us a norm

$$\|g\|_\phi = \|\phi(\cdot) - \phi(g^{-1}\cdot)\|_\infty = \|\partial_g\phi\|_\infty,$$

and a corresponding left-invariant metric

$$d_\phi(g,h) = \|g^{-1}h\|_\phi.$$

Note that since ψ is Lipschitz we have

$$\|\partial_g\psi\|_\infty = O(\|g\|)$$

As a step towards the commutator estimate, let us show that

$$\|\partial_g\partial_h\phi\|_\infty = O(\|g\|\|h\|) \tag{10.1}$$

whenever $g,h \in B(1,\epsilon)$.

Note that by the left invariance of the Haar measure one has

$$\begin{aligned}\tau(h)(F * G)(x) &= \int F(y)G(y^{-1}h^{-1}x)d\mu(y)\\ &= \int F(h^{-1}y)G(y^{-1}x)d\mu(y)\\ &= (\tau(h)F) * G(x).\end{aligned}$$

One also has

$$\tau(g)(F * G)(x) = \int F(y)G(y^{-1}g^{-1}x)d\mu(y) = \int F(y)(\tau(g^y)G(y^{-1}x))d\mu(y)$$

where $g^y = y^{-1}gy$.

Combining the two in our case, one gets

$$\partial_g\partial_h\psi(x) = \int (\partial_h\psi)(y)(\partial_{g^y}\psi)(y^{-1}x)d\mu(y).$$

When $h \in B(1,\epsilon)$, the integrand is nonzero only when $y \in B(1,2\epsilon)$, and since ψ is Lipschitz continuous we have

$$\|\partial_g\partial_h\psi\|_\infty = O(\|h\| \sup_{y\in B(1,2\epsilon)} \|g^y\|)$$

It suffices to show that $\|g^y\| = O(g)$ for $y \in B(1, 2\epsilon)$. This follows from the escape property in the following way. Let n be such that $n\|g\| \le \epsilon$ then $\|g^n\| \le \epsilon$, and $\|(g^y)^n\| = \|(g^n)^y\| \le 5\epsilon$, and by the escape property $\|g^y\| = O(\frac{1}{n})$.

Now, by (10.1) we get that for small enough g, h

$$\begin{aligned}\|[g,h]\|_\phi &= \|\tau([g,h])\phi - \phi\|_\infty \\ &= \|\tau(g)\tau(h)\phi - \tau(h)\tau(g)\phi\|_\infty \\ &= \|\partial_g\partial_h\phi - \partial_h\partial_g\phi\|_\infty = O(\|g\|\|h\|)\end{aligned}$$

It remains to prove $\|x\| = O(\|x\|_\phi)$ for small enough x. Let n be such that $n\|x\|_\phi < \|\phi\|_\infty$. And so $\|x^n\|_\phi < \|\phi\|_\infty$. This implies that ϕ and $\tau(x^n)\phi$ have overlapping supports, and thus $x^n \in B(1, 4\epsilon)$, which by the escape property implies $\|x\| = O(\frac{1}{n})$, as desired. □

□

11 No Small Subgroups and Escape Norms

Definition 11.1. A topological group G has *no small subgroup (NSS)* if there exists an 1 neighborhood that does not contain any non-trivial subgroups.

Remark 11.2. Note that a Lie group has the NSS property, as locally in radially homogeneous coordinates, any cyclic group must escape some neighborhood of the identity element.

Our goal is to prove the converse.

Theorem 11.3. *A locally compact group is NSS if and only if it is isomorphic to a Lie group.*

To appreciate this Theorem, let us observe the following immediate corollary.

Corollary 11.4. *A closed subgroup of a Lie group is a Lie group.*

Proof. A closed subgroup of a locally compact group with the NSS property, is a locally compact group with the NSS property □

Lemma 11.5. *Let G be an NSS group. Let U be a symmetric precompact open identity neighborhood such that $\overline{U}$ has no non-trivial subgroups. Then the open sets*

$$U_{[N]} = \{g \in G \,:\, g, g^2, \dots, g^N \in U\}$$

form a local basis of open 1*-neighborhoods for G.*

Proof. $U_{[N]}$ is clearly open. Assume $g_n \in U_n$ are such that $g_n \notin V$ for some open 1-neighborhood. Then, from compactness of $\overline{U}$, we may assume that $g_n \to g \in \overline{U} - V$. Then, $g^n \in \overline{U}$ for all n, contradicting the assumption on $\overline{U}$. □

Proof of Theorem 11.3 (minus a technical lemma). As we explained in Remark 11.2, one implication is obvious. To prove the other implication we will show that an NSS locally compact group admits a weak Gleason metric, which then, by Theorem 10.2 we will be done.

Let U_0 be a symmetric precompact open 1-neighborhood whose closure does not contain any non-trivial subgroups. Define the *escape norm* of U_0, by

$$\|g\|_{U_0} = \inf\left\{\frac{1}{n+1} \,:\, g \in U_{0[n]}\right\}.$$

Strictly speaking it is not a group norm (as it does not satisfy the triangle inequality), but since U_0 does not have any non-trivial subgroup, it is positive, i.e $\|g\|_{U_0} > 0$ for $1 \neq g \in G$.

By Lemma 11.5, if we replace U_0 by another symmetric precompact open 1-neighborhood V_0 whose closure does not contain any non-trivial subgroups, we get a 'bi-Lipschitz' equivalent escape norm. That is,

$$\|g\|_{V_0} \ll \|g\|_{U_0} \ll \|g\|_{V_0}. \tag{11.1}$$

To force the escape norm to satisfy the triangle inequality we modify by taking an infimum

$$\|g\|_{*,U_0} = \inf\left\{\sum_{i=1}^{n} \|g_i\|_{U_0} \ : \ g = g_1,\ldots,g_n\right\}.$$

Clearly $\|\cdot\|_{*,U_0}$ is a pseudo-norm, i.e,

$$\|gh\|_{*,U_0} \le \|g\|_{*,U_0} + \|h\|_{*,U_0} \text{ and } \|g\|_{*,U_0} = \|g^{-1}\|_{*,U_0}.$$

However, when taking the infimum we might have lost its positivity. For this we will need the following Lemma which can be seen as a quasi-triangle inequality for $\|\cdot\|_{U_0}$.

Lemma 11.6. *For any n and any $g_1,\ldots,g_n \in G$, one has*

$$\|g_1\cdots g_n\|_{U_0} \le M \sum_{i=1}^{n} \|g_i\|_{U_0}$$

(where the constant M depends only on U_0).

We postpone the proof of this lemma, and continue with the proof of Theorem 11.3

Lemma 11.6 implies,

$$\frac{1}{M}\|g\|_{U_0} \le \|g\|_{*,U_0} \le \|g\|_{U_0}.$$

In other words, $\|\cdot\|_{U_0} \asymp \|\cdot\|_{*,U_0}$ are bi-Lipschitz equivalent. In particular, $\|g\|_{*,U_0}$ is positive for $1 \ne g \in G$, and therefore $\|\cdot\|_{*,U_0}$ defines a left-invariant metric by

$$d_{*,U_0}(g,h) = \|g^{-1}h\|_{*,U_0}.$$

By Lemma 11.5, the escape-norm 'balls' $\{g \ : \ \|g\|_{U_0} < \epsilon\}$ form a local basis for the topology on G. Therefore, the norm $\|\cdot\|_{*,U_0}$ generates the topology on G.

To show that $\|\cdot\|_{*,U_0}$ defines a weak Gleason metric, it remains to show that it has the escape property. That is, there exists $\epsilon > 0$ such that if $g \in G, n \in \mathbb{N}$ are such that $n\|g\|_{*,U_0} < \epsilon$, then $\|g^n\|_{*,U_0} \gg n\|g\|_{*,U_0}$.

Let U_1 be an identity neighborhood such that $U_1^2 \subseteq U_0$. Let $\epsilon > 0$ be such that $\|h\|_{*,U_0} < \epsilon \implies h \in U_1$. If $n\|g\|_{*,U_0} < \epsilon$ then by the triangle inequality for all $i = 1,\ldots,n$, $\|g^i\|_{*,U_0} < \epsilon$ and hence $g^i \in U_1$.

Hence, if $n\|g\|_{*,U_0} < \epsilon$ and $1, g^n, g^{2n}, \ldots, g^{nm} \in U_1$ then $1, g, g^2, \ldots, g^{nm} \in U_1^2 \subseteq U_0$. Therefore, if $n\|g\|_{*,U_0} < \epsilon$, $\|g^n\|_{U_1} \ge n\|g^n\|_{U_0}$. The claim follows since $\|\cdot\|_{U_1}, \|\cdot\|_{U_0}, \|\cdot\|_{*,U_0}$ are bi-Lipschitz equivalent.

This concludes the proof of Theorem 11.3. □

Proof of Lemma 11.6. The proof of the lemma is a bit weird. We start by assuming that the desired inequality

$$\|g_1 \cdots g_n\|_{U_0} \le M \sum_{i=1}^{n} \|g_i\|_{U_0} \tag{11.2}$$

holds for some M, and prove that M could be improved to some constant $O(1)$ that does not depend on the initial M. After we are done proving this, we will see how to tweak the proof so that some tweaked version of (11.2) obviously holds, but which would still allow for the desired conclusion.

Note that under this assumption $\|g\|_{*,U_0}$ is a well defined norm, and by (11.2) satisfies

$$\frac{1}{M}\|g\|_{U_0} \le \|g\|_{*,U_0} \le \|g\|_{U_0},$$

for some M.

Let us $\psi : G \to \mathbb{R}$ be the Lipschitz function defined by

$$\psi(x) = \max\{0, 1 - M d_{*,U_0}(x, U_0)\}$$

On the one hand, it satisfies

$$|\partial_g \psi(x)| \le M \|g\|_{U_0} \tag{11.3}$$

On the other, it is supported on

$$\{x : d_{*,U_0}(x, U_0) \le M\} \subseteq \{x : d_{U_0}(x, U_0) \le 1\} \subseteq U_0^2.$$

Let L be a large number, to be chosen later, and let U_1 be a small 1-neighborhood depending on L, to be chosen later. Define $\eta : G \to \mathbb{R}$ by

$$\eta(x) = \sup\left\{1 - \frac{j}{L} : x \in U_1^j U_0\right\} \cup \{0\}.$$

For all $g \in U_1$ and $x \in G$, it satisfies

$$|\partial_g \eta(x)| \le \frac{1}{L}, \tag{11.4}$$

and η is supported in $U_1^L U_0$, which by choosing U_1 small enough, we may assume $U_1^L U_0 \subseteq U_0^2$.

η and ψ are bounded, compactly supported, Borel measurable functions. Therefore, their convolution $\phi = \psi * \eta$ is well defined, continuous, supported on U_0^4 and satisfies $\phi(1) = \mu(U_0) \gg 1$.

It follows that if $n\|g\|_\phi < \phi(0)$, then $\|g^n\|_\phi < \phi(0) \implies g^n \in U_0^8$. Hence,

$$\|g\|_{U_0^8} \ll \|g\|_\phi$$

for all $g \in G$, and therefore by (11.1) also

$$\|g\|_{U_0} \ll \|g\|_\phi. \tag{11.5}$$

As in the proof of Proposition 10.15, we can write

$$\partial_g \partial_h \phi(x) = \int_G (\partial_h \psi)(y)(\partial_{g^y} \eta)(y^{-1}x) d\mu(y).$$

If $h \in U_0$ the integrand vanishes unless $y \in U_0^3$. Let $U_2 \subseteq U_1$ be a small 1-neighborhood such that $g^y \in U_1$ for all $g \in U_2$ and $y \in U_0^3$. Then, if $h \in U_0$ and $g \in U_2$ by (11.3) and (11.4) we have

$$|\partial_g \partial_h \phi(x)| \ll \frac{M}{L} \|h\|_{U_0}. \tag{11.6}$$

By the identity

$$\partial_{g^n} = n\partial_g + \sum_{i=0}^{n-1} \partial_{g^i} \partial_g$$

and triangle inequality we have

$$\|g^n\|_\phi = n\|g\|_\phi + O\left(\sum_{i=0}^{n-1} \|\partial_{g^i} \partial_g \phi\|_\infty\right)$$

By the inequality (11.6) we conclude

$$\|g^n\|_\phi = n\|g\|_\phi + O\left(n\| \frac{M}{L} \|g\|_{U_0}\right)$$

whenever $g, \dots, g^n \in U_2$. Using some bound $\|g^n\|_\phi = O(1)$ we get

$$\|g\|_\phi \ll \frac{1}{n} + n\frac{M}{L} \|g\|_{U_0}$$

Optimizing over n we get

$$\|g\|_\phi \ll \|g\|_{U_2} + \frac{M}{L} \|g\|_{U_0}$$

And by (11.1),

$$\|g\|_\phi \ll \left(\frac{M}{L} + O_{U_2}(1)\right) \|g\|_{U_0}.$$

This, together with the triangle inequality

$$\|g_1 \dots g_n\|_\phi \le \sum_{i=1}^{n} \|g_i\|_\phi$$

and (11.5), gives

$$\|g_1 \dots g_n\|_{U_0} \ll \left(\frac{M}{L} + O_{U_2}(1)\right) \sum_{i=1}^{n} \|g_i\|_{U_0}.$$

By choosing L large enough, and iterating we get

$$\|g_1 \dots g_n\|_{U_0} \ll C \sum_{i=1}^{n} \|g_i\|_{U_0}$$

for some constant C.

Now, to actually prove the lemma, one cannot assume *a priori* that such a bound exists. Instead, one destroys slightly the escape norm by adding an $\epsilon > 0$ to it. This automatically implies a constant M in (11.2) of order $\approx 1/\epsilon$. One repeats the argument above to show

$$\|g_1 \dots g_n\|_{U_0} \ll C \sum_{i=1}^{n} (\|g_i\|_{U_0} + \epsilon)$$

with C which does not depend on ϵ, and then take $\epsilon \to 0$ to complete the proof. □

12 Subgroup trapping and the Gleason-Yamabe Theorem

12.1 The subgroup trapping property

Definition 12.1. For a 1-neighborhood V in a topological group G, an element g is *trapped* in V if $\langle g \rangle \subseteq V$. We denote by $Q[V]$ the set of trapped elements, or equivalently the union of all subgroups contained in V. A topological group has the *subgroup trapping property*, if every 1-neighborhood U contains a 1-neighborhood V, such that $\langle Q[V] \rangle \subseteq U$.

Theorem 12.2 (weak Gleason-Yamabe for groups with subgroup trapping)**.** *Let G be a locally compact group with the subgroup trapping property, and let U be an open 1-neighborhood, then there exists $G' \leq G$ open, and a compact normal subgroup $N \lhd G'$ contained in U, such that G'/N is Lie.*

Proof. By Theorem 11.3 it suffices to show that we can choose G' and N as above such that G'/N is a locally compact group with NSS.

Let G and U be as in the theorem, by decreasing U we may assume that U is compact. From the subgroup trapping property there exists a 1-neighborhood $V \subseteq U$ such that $\langle Q[V] \rangle \subseteq U$. Hence, $H = \overline{\langle Q[V] \rangle} \subseteq U$ is a compact group.

We can thus apply the Gleason-Yamabe Theorem for compact groups (Corollary 5.8) to H. Therefore, there exists a compact normal subgroup $N \lhd H$ which is contained in V, such that H/N is a Lie group, and in particular has NSS.

Let W be a small identity neighborhood such that:

1. $WNW \subseteq V$ (can be achieved by compactness of N)

2. $(WNW \cap H)/N$ is a 1-neighborhood of H/N which does not contain non-trivial subgroups in H/N (can be achieved since H/N has NSS).

Let $G' = \langle W, N \rangle$. Clearly, G' is open in G.

To prove that $N \lhd G'$, let $g \in W$. Since $N^g \subseteq WNW \subseteq V$ we get that $N^g \in Q[V]$ hence $N^g \in H$. Therefore, $N^g \subseteq (WNW \cap H)$. By the second property above $N^g \leq N$. Therefore, $N \lhd G'$

To prove that G'/N has NSS, observe that if $K \subseteq WNW/N$ is a subgroup of G'/N, then $KN \subseteq WNW \subseteq V$ is a subgroup in G, hence $KN \subseteq Q[V]$, hence $KN \subseteq H$, from which it follows that KN/N is a subgroup of H/N which is contained in $(WNW \cap H)/N$, contradicting the second property above. □

12.2 Weak Gleason-Yamabe Theorem

Theorem 12.3 (Weaker version of Gleason-Yamabe)**.** *Let G be a locally compact group, and let U be a 1-neighborhood in G. Then there exist $G' \leq G$ an open subgroup, and $N \lhd G'$ contained in U, such that G'/N is a Lie group.*

By Theorem 12.2, it suffices to show the following.

Proposition 12.4. *Every locally compact metrizable group has the subgroup trapping property.*

In order to prove this proposition, we will need the following lemma which controls the growth rate of subsets of $Q[V]$ for small enough V.

Lemma 12.5 (Finite trapping)**.** *Let G be a locally compact group, let U be an open precompact 1-neighborhood, and let $m \geq 1$. Then there exists a 1-neighborhood V such that if $Q \subseteq Q[V]$ is a symmetric set containing the identity satisfying that $Q^n \subseteq U$ then $Q^{nm} \subseteq U^8$.*

We will postpone the proof of this lemma, and first prove the proposition.

In the proof we will encounter a limit of sets.

Definition 12.6. Let (X, d) be a metric space, and let A, B be compact subsets of X. The Hausdorff distance of A, B is defined as

$$\begin{aligned} d_H(A, B) &= \inf\{\epsilon | A_\epsilon \supset B \text{ and } B_\epsilon \supset A\} \\ &= \|d(A, \cdot) - d(B, \cdot)\|_\infty \end{aligned}$$

where A_ϵ, B_ϵ denote the ϵ-neighborhoods of A, B respectively.

Exercise 12.7. If X is a compact metric space. Then the space of all non-empty closed subsets of X is compact with respect to the the Hausdorff distance. (E.g use Arzela-Ascoli on the space $\{d(A, \cdot)\}_{A \subseteq X}$)

Proof of Proposition 12.4. Let G be a locally compact metrizable group, and let U be a compact 1-neighborhood. Assume for contradiction, that G does not have the subgroup trapping property. If V_i denotes the ball of radius $1/i$ around 1 in G, then by assumption $\langle Q[V_i] \rangle \nsubseteq U$. Therefore, there exists a minimal n_i such that $Q[V_i]^{n_i+1} \nsubseteq U$ (and $Q[V_i]^{n_i} \subseteq U$). Note that $n_i \to \infty$ since V_i converge to 1.

Therefore, by Lemma 12.5, we can find a sequence $m_i \to \infty$ such that $Q[V_i]^{m_i n_i} \subseteq U^8$.

$\overline{Q[V_i]}_i^n$ are closed subsets of the compact metric space $\overline{U}$, therefore, they have a converging subsequence (with respect to the Hausdorff metric) to some $E \subseteq \overline{U}$. Since for all m, $\overline{Q[V_i]}^{mn_i} \to E^m$ and $\overline{Q[V_i]}^{mn_i} \subseteq \overline{U^8}$, we see that $H = \overline{\langle E \rangle}$ is a compact subgroup of G contained in $\overline{U^8}$.

Let U' be a 1-neighborhood of G, to be chosen later. By the Gleason-Yamabe Theorem for compact groups (Corollary 5.8) applied to H, there exists a compact normal subgroup $N \lhd H$ contained in U', such that H/N is a Lie group. Let $\pi : H \to H/N$ be the quotient map. In particular, there exists some 1-neighborhood $B \subseteq H/N$ such that B^{10} does not contain non-trivial subgroups of H/N. By compactness of $H/N - B$, we see that there exists k such that if $g \in H/N - B$ then one of $1, g, \dots g^k$ is not in B^{10}.

We will derive a contradiction by showing that $\overline{\pi^{-1}(B^8)}$ and $H - \pi^{-1}(B^{10})$ are arbitrarily close in G, while they are close and disjoint.

Now, for $\epsilon > 0$ small enough, let us consider the ϵ-neighborhood $\pi^{-1}(B)_\epsilon$ of $\pi^{-1}B$. Assume that ϵ is small enough so that $\pi^{-1}(B)_\epsilon \subseteq U$. Since $Q[V_i]^{n_i+1} \nsubseteq U$, then $Q[V_i]^{n_i+1} \nsubseteq \pi^{-1}(B)_\epsilon$. Let $n_i' \le n_i$ be the smallest such that $Q[V_i]^{n_i'+1} \nsubseteq \pi^{-1}(B)_\epsilon$. In particular, $Q[V_i]^{n_i'} \subseteq \pi^{-1}(B)_\epsilon$, and $n_i' \to \infty$.

Using Lemma 12.5 again, we see that $Q[V_i]^{n_i' m} \subseteq (\pi^{-1}(B)_\epsilon)^8$ for all m as $i \to \infty$.

On the other hand, $Q[V_i]^{n_i'+1}$ converges to a subset of H in the Hausdorff metric (since $Q[V_i]^{n_i} \to\subseteq H \implies Q[V_i]^{2n_i} \to\subseteq H \implies Q[V_i]^{n_i'+1} \to\subseteq H$ as $n_i' \le n_i$). Let $g_i \in Q[V_i]^{n_i'+1} - \pi^{-1}(B)_\epsilon$, then for large enough i g_i is ϵ-close to some $h_i \in H$. It follows that $h_i \notin \pi^{-1}(B)_\epsilon$, and hence $h_i^{j_i} \notin \pi^{-1}(B^{10})$ for some $1 \le j_i \le k$. But then, $g_i^{j_i} \in Q[V_i]^{k(n_i'+1)}$ and hence in $(\pi^{-1}(B)_\epsilon)^8$. It follows that $h_i^{j_i} \in H - \pi^{-1}(B^{10})$ is arbitrarily close to some elements of $\overline{\pi^{-1}(B^8)}$, as desired. □

Proof of Lemma 12.5. Assume U, m as in the lemma. Let V be a small neighborhood to be chosen later, and let $Q \subseteq V$ such that $Q^n \subseteq U$. Our goal is to show that $Q^{mn} \subseteq U^8$.

The idea of the proof is to find a function ϕ (using convolution of 'Lipschitz' functions) that depends on some large parameter M (to be chosen later) and show that $\|q\|_\phi \ll \frac{1}{nM}$ for all $q \in Q$. By the triangle inequality it implies that $\|g\|_\phi \ll \frac{m}{M}$ for all $g \in Q^{mn}$, and the claim will follow by choosing M large enough so that this inequality will imply that $g \in U^8$.

Let $\psi : G \to [0,1]$ be the function defined by

$$\psi(x) = \sup\{1 - \frac{j}{n} : x \in Q^j U\} \cup \{0\}.$$

Observe that ψ is supported in U^2, and obeys

$$\|\partial_q \psi\|_\infty \le \frac{1}{n}$$

for $q \in Q$. The second function $\eta : G \to \mathbb{R}$ is defined by

$$\eta(x) = \sup\{1 - \frac{j}{M} : x \in (V^{U^4})^j U\} \cup \{0\}.$$

where M will be chosen later (as explained above). By choosing V small enough we assume $(V^{U^4})^M \subset U$, hence also η is supported in U^2, and satisfies

$$\|\partial_g \eta\|_\infty \le \frac{1}{M}$$

for all $g \in V^{U^4}$.

Let $\phi = \psi * \eta$. It is thus supported on U^4 and $\|\phi\|_\infty \gg 1$ (independent on M and V). As before, if $\|g\|_\phi$ is sufficient small (e.g $< \|\phi\|_\infty$) then $g \in U^8$.

Now let us estimate $\|q\|_\phi$ for $q \in Q \subseteq Q[V]$. Recall from the proof of Proposition 11.6 the identity

$$\partial_{q^n} = n\partial_q + \sum_{i=0}^{n-1} \partial_{q^i} \partial_q$$

and therefore

$$n\|q\|_\phi \ll \|q^n\|_\phi + \sum_{i=0}^{n-1} \|\partial_{q^i}\partial_q \phi\|_\infty$$

hence

$$\|q\|_\phi \ll \frac{1}{n}\|q^n\|_\phi + \sup_{i=0,\dots,n-1} \|\partial_{q^i}\partial_q \phi\|_\infty.$$

We have seen that each of the expressions can be written as

$$\partial_{q^i}\partial_q\phi(x) = \int_G (\partial_q \psi)(y)(\partial_{(q^n)^y}\eta)(y^{-1}x)d\mu(y)$$

Note that $y \in U^4$ for it to be non trivial, and $q^n \in V$ since $q \in Q[V]$. Therefore, we may use our 'Lipschitz' estimates to conclude

$$|\partial_{q^i}\partial_q\phi(x)| \ll \frac{1}{Mn}.$$

Similarly, one estimates

$$\|q^n\|_\phi \ll \frac{1}{M},$$

and together they give

$$\|q\|_\phi \ll \frac{1}{nM}.$$

Now, for all $g \in Q^{mn}$

$$\|g\|_\phi \ll \frac{m}{M}.$$

and by choosing M sufficiently large, we can guarantee that $g \in U^8$, as desired. □

12.3 Strong Gleason-Yamabe

Theorem 12.8 (Gleason-Yamabe)**.** *Let G be a locally compact group. Then there exists $G' \leq G$ open such that every 1-neighborhood U contains a compact normal subgroup $N \lhd G'$ such that G'/N is Lie.*

In particular, if G is a Hausdorff locally compact group, then there exists $G' \leq G$ open such that G' is an inverse limit of Lie groups.

Proof. In Proposition 3.2 we saw that the connected component G_0 of $1 \in G$ is a closed normal subgroup and $H = G/G_0$ is totally disconnected. Let us denote by $\pi : G \to H$ the quotient map. By van-Danzig's Theorem (Theorem 3.4), H contains a compact open subgroup $H' \leq H$. Let $G' = \pi^{-1}(H') \leq G$ be the open subgroup pullback of H'. Then G' is connected-by-compact.

By the weak Gleason-Yamabe Theorem (Theorem 12.3), for every $1 \in U$ open, there is an open subgroup $G'' \leq G'$ and a normal subgroup $N'' \lhd G''$ contained in U such that G''/N'' is Lie. Since G'' is open and G_0 is connected, we must have $G_0 \leq G''$, hence $G'' = \pi^{-1}(H'')$ where $H'' = \pi(G'') \leq H'$.

Since H'' is open and H' is compact, it must have finite index $[H' : H''] < \infty$ and therefore $[G' : G''] < \infty$. Let $g_1, \ldots, g_k \in G'$ be coset representatives of G''. Hence $N' = \mathrm{Core}_{G'}(N'') = N''^{g_1} \cap \ldots \cap N''^{g_k}$. Clearly, $N' \lhd G'$ is a compact and contained in U. It remains to show that G'/N' is Lie.

By Theorem 11.3, it suffices to show that G'/N' has NSS. By the NSS property of G''/N'', there exists $1 \in V \subseteq G'$ open, such that every subgroup of V is contained in N''. By intersecting $W = \bigcap_{i=1,\ldots,k} V^{g_i}$ we may assume that every subgroup of W is contained in N''^{g_i} for all $i = 1, \ldots, k$, and thus contained in N'. This completes the proof. ☺ □

13 Hilbert's Fifth Problem and Beyond

13.1 Hilbert's Fifth Problem

Theorem 13.1 (Solution to Hilbert's Fifth Problem). *Let G be a locally compact topological group which is locally Euclidean, then G is Lie.*

Proof. Assume that G is locally homeomorphic to $\mathbb{R}^d$. Clearly, G is Hausdorff and first countable. Moreover, G_0 is open, since it suffices to show that an open subgroup is Lie, by passing to G_0 we may assume that G is connected. By Theorem 12.8 G is an inverse limit $\varprojlim G_n$ of a sequence of Lie groups,

$$\ldots \to G_{n+1} \to G_n \to \ldots \to 1.$$

Each $G_n = G/N_n$, where $N_n \lhd G$ are compact. Let us denote by $M_n = N_n/N_{n+1} \lhd G_{n+1}$, then $G_{n+1}/M_{n+1} = G_n$.

Note that the map $q_n : G_{n+1} \to G_n$ gives a linear map between the finite dimensional vector spaces (Lie algebras) $L(q_n) : L(G_{n+1}) \to L(G_n)$ which must be surjective (since exp is a local homeomorphism, and the quotient map is a local surjection). We can therefore find a linear injective section $T_n : L(G_n) \to L(G_{n+1})$ such that $L(q_n) \circ T_n = \mathrm{id}_{L(G_n)}$

If $\phi \in L(G_n)$ is a 1-parameter subgroup of G_n, then $T_n(\phi)$ is a 1-parameter subgroup which is admissible with q_n. i.e $q_n(T_n(\phi))(t) = \phi(t)$. Therefore, the sequence $T_{n,m}(\phi) \coloneqq T_m \circ \ldots \circ T_n(\phi)$ for all $m \geq n$ is an admissible sequence of 1-parameter subgroups. Which then give a 1-parameter subgroup $T_{n,\infty}(\phi)$. The map $T_{n,\infty} : L(G_n) \to L(G)$ is injective and continuous (since it has a left inverse $L(G \to G_n)$).

Therefore, $\psi_n : \exp \circ T_{n,\infty}(\phi) \circ \exp^{-1} : U_n \to U$ is a continuous injective map from a 1-neighborhood U_n in G_n to a 1-neighborhood U in G, which is a local inverse to the quotient map $Q_n : G \to G_n$. We may assume that U_n and U are homeomorphic to $\mathbb{R}^{d_n}$ and $\mathbb{R}^d$. It follows that $d_n \leq d$.

Therefore, the dimension of G_n must stabilize. Without loss of generality, $\dim(G_n) = \dim(G_{n+1}$ for all n. It follows that $\dim(M_n) = 0$ for all n, hence M_n is finite (it is a compact Lie group of dimension 0). Therefore, N_1/N_n are finite. Since $N_1 = \lim_n N_1/N_n$, we get that N_1 is a profinite group, and in particular totally disconnected.

We claim that the continuous injective map $\psi = \psi_1 : U_1 \to U$ is a homomorphism. Clearly $\psi(1) = 1$ and $\psi(g)^{-1} = \psi(g^{-1})$. Let V be a small symmetric connected open neighborhood in G_1, such that $V^3 \subseteq U_1$. Then $F = \{\psi(g)\psi(h)\psi((gh)^{-1}) \mid g,h \in V\} \subseteq G$ is contained in N_1 (because ψ is a local inverse to Q_1), it contains 1, and connected. Since N_1 is totally disconnected $F = \{1\}$.

Similarly, $\psi(V)$ commutes with N_1. (For all $k \in N_1$ consider the set

$$F_k = \{\psi(g)^{-1}k\psi(g) \mid g \in V\} \subseteq G)$$

and apply a similar argument). Hence G is locally isomorphic to the local group $V \times N_1$.

However, G is locally connected, hence N_1 must be discrete, and G is therefore locally isomorphic to the Lie local group V. By Theorem 8.11 we are done. ☺! □

13.2 Hilbert-Smith Conjecture

Conjecture 13.2 (The Hilbert-Smith Conjecture)**.** *Let G be a locally compact topological group, and let M be a connected (finite dimensional) manifold. If $G \curvearrowright M$ continuously and faithfully then G is a Lie group.*

Note that the all assumptions are needed:

1. If G is not locally compact, then we can take $G = \mathrm{Diff}(\mathbb{S}^1) \curvearrowright \mathbb{S}^1$.
2. If M is not connected, then we can take $(\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \curvearrowright \mathbb{R}/\mathbb{Z} \times \mathbb{N}$.
3. If the action is not continuous/faithful then we can easily find counter examples.

Note also that the Hilbert-Smith Conjecture is a generalization of Hilbert's Fifth Problem. (If G is locally euclidean, then we can look at $G_0 \curvearrowright G_0$, and deduce by the conjecture that G_0 is Lie, and hence G is Lie).

Remark 13.3. Using the Gleason-Yamabe Theorem one can deduce that if a counter example to the conjecture exists then there is a counter example of the form $\mathbb{Z}_p \curvearrowright M$.

Using this, the conjecture was proven when $\dim(M) \leq 3$ (The case ≤ 2 was done by Montgomery-Zippin, and the case $= 3$ was done by Pardon in 2013).