4 Free splittings and ends of groups

4.1 Free splittings and Grushko's Theorem

Our first goal is to study the simplest possible splittings of groups, those are splittings over trivial groups. Such a splitting is called a free splitting: a free product with trivial amalgamation is simply a free product, and an HNN over the trivial group is a free product with an infinite cyclic group. We say that a group is *freely indecomposable* if whenever $G = A * B$ then either $A = 1$ or $B = 1$.

Theorem 4.1 (Grushko)**.** *Let G be a f.g group, then there exist freely indecomposable subgroups $A_1, \ldots, A_m$ and r, such that $G \simeq A_1 * \ldots * A_m * F_r$. Moreover, this decomposition is unique up to reordering (and conjugacy) of the groups A_i.*

Lemma 4.2. *Let $G = A*B$, and let $f : F \to G$ be a surjective map from a free group F to G. Then, there exists a free decomposition $F = F_A * F_B$ such that $f(F_A) = A$ and $f(F_B) = B$. In particular, $r(G) = r(A) + r(B)$, where $r(G) = \min\{|S| \mid G = \langle S \rangle\}$ is the minimal size of generating set for G.*

Stallings' proof. @ □

Exercise 4.3. Deduce Grushko's Theorem from the lemma.

Theorem 4.4. *(Kurosh) Let $H \leq A*B$ then H is the fundamental group of graph of groups in which all the vertex groups are conjugate into A or B.*

Exercise 4.5. Prove Kurosh's Theorem.

4.2 Ends of spaces and groups

Definition 4.6. Let X be a locally finite[7] graph. Then the *number of ends* of X is defined as the supremum of the number of infinite (or equivalently, unbounded) components obtained after removing a finite set of vertices.

Example 4.7. The infinite line graph $\cdots\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\cdots$ has 2 ends

We will mostly be interested in graphs which come from groups. Recall that if G is a group which is generated by a finite subset $S \subseteq G$, then *the Cayley graph of G with respect to S* is the locally finite connected graph $\mathrm{Cay}(G, S)$ whose vertices are G and has an edge connecting g and g' if $g' = gs$ for some $s \in S$.

Definition 4.8. The *number of ends* of a finitely generated group G is the number of ends of a Cayley graph $\mathrm{Cay}(G, S)$ for some finite generating set S of G.

Exercise 4.9. Show that the number of ends of a group does not depend on the generating set.

Example 4.10. Because of the previous example we see that the number of ends of the group $\mathbb{Z}$ is 2. What is the number of ends of $\mathbb{Z}^2$? of a free group F_2?

Exercise 4.11.
1. Show that the number of ends of a group is 0, 1, 2, or ∞. We call the group $0/1/2/\infty$-ended respectively.

[7] this means that every vertex is incident to finitely many edges

2. Show that if G has a splitting over a finite group then G has more than one end.

3. Show that if G is 2-ended then it contains a finite-index cyclic subgroup.

Theorem 4.12 (Stallings' theorem on ends of groups)**.** *If a finitely generated group G has more than one end, then G splits over a finite group.*

Dunwoody's proof. @ □

Theorem 4.13 (Dunwoody's accessibility)**.** *Every finitely presented group has a splitting over finite edge groups so that all the vertices are either finite or 1-ended. Moreover, the 1-ended pieces are unique.*

Proof. @ □