1 Complexes, homology and cohomology

1.1 Simplicial complexes

Definition 1.1. Let V be a non-empty set. Let $\mathcal{P}_{\text{fin}}(V)$ be the collection of finite non-empty subsets of V. An *(abstract) simplicial complex* is a set $\Sigma \subseteq \mathcal{P}_{\text{fin}}(V)$ such that all singletons of V are in Σ and if $\tau \subseteq \sigma \in \Sigma$ then $\tau \in \Sigma$. We will denote $\dim(\sigma) = \#\sigma - 1$, and denote by $\Sigma^k = \{\sigma \in \Sigma \mid \dim \sigma = k\}$. An element of Σ^k is called a *k-simplex.*

Given a simplicial complex Σ its *realization* $|\Sigma|$ is the topological space

$$|\Sigma| = \{f \in [0,1]^V : \operatorname{supp} f \in \Sigma \wedge \sum f = 1\}$$

with the topology $A \subseteq |\Sigma|$ is open if and only if $A \cap [0,1]^\sigma$ is open in $[0,1]^\sigma$ for all $\sigma \in \Sigma$ (with the standard topology). Each k-simplex σ gives rise to a (geometric) simplex $|\Sigma| \cap [0,1]^\sigma$ in $|\Sigma|$ which is homeomorphic to $\mathbb{D}^k$. We will usually consider V, Σ to be finite, in which case $|\Sigma|$ can simply given the subset topology from $[0,1]^V$.

Example 1.2. If $V = a, b, c, d, e$ and Σ is the smallest simplicial complex containing the subsets $\{a,c\}, \{b,c\}, \{b,d\}, \{c,d,e\}$ (I.e., Σ contains those sets and all of their subsets). Then $|\Sigma|$ is homeomorphic to the space shown in Figure 1.

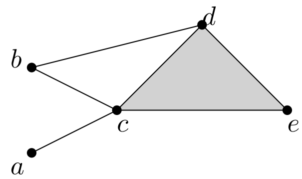

Figure 1: $|\Sigma|$

Exercise 1.3. Find a simplicial complex whose realization is homeomorphic to the 2-sphere $\mathbb{S}^2$, the 2-torus $\mathbb{T}^2$, the projective plane $\mathbb{P}^2$...

Show that the realization of a simplicial complex is naturally a CW complex.

1.2 Homology

Throughout, let $\mathbf{F} = \mathbf{F}_2 = \{0,1\}$ the field with two elements.

Definition 1.4 (Simplicial chain complex over $\mathbf{F}$)**.** Let Σ be a simplicial complex, consider the vector space $C_k(\Sigma) = \operatorname{Span}_{\mathbf{F}}(\Sigma^k)$, elements in C_k are called k-chains, and are formal finite sums of simplices of dimension k with coefficients in $\mathbf{F}$. We will often consider them simply as finite subsets of Σ^k. Consider the linear map $\partial_k : C_k(\Sigma) \to C_{k-1}(\Sigma)$ defined on the basis elements $\sigma \in \Sigma^k$ by

$$\partial_k \sigma = \sum_{\sigma \supset \tau \in \Sigma^{k-1}} \tau.$$

The sequence of spaces and maps (C_k, ∂_k) is called the *chain complex of* Σ. Elements of C_k are called *k-chains.*

Exercise 1.5. Verify that $\partial^2 = 0$.

Definition 1.6 (Cycles, boundaries and homology)**.** Denote by $Z_k(\Sigma) = \ker \partial_k$, and call its elements *k-cycles.* Denote by $B_k(\Sigma) = \operatorname{im} \partial_{k+1}$, and call its elements *k-boundaries.*

Then the k-th *homology* of Σ (with coefficients in $\mathbf{F}$) is defined as the vector space quotient

$$H_k(\Sigma) = Z_k(\Sigma)/B_k(\Sigma).$$

Note that the previous exercise shows that indeed $B_k \subseteq Z_k$.

Fact 1.7. $H_k(\Sigma)$ is a homotopy equivalence invariant of the space $|\Sigma|$ and does not depend on the specific combinatorial structure of Σ. We will therefore simply denote $H_k(X)$ for a topological space which is homeomorphic to a realization of a simplicial complex.

Exercise 1.8. If Σ is finite, then its *Euler characteristic* is defined by $\chi(\Sigma) = \sum_i (-1)^i \#\Sigma^i$. Show that

$$\chi(\Sigma) = \sum_i (-1)^i \dim_{\mathbf{F}_2}(H^i(\Sigma))$$

Let us try to understand the homology a bit better.

Exercise 1.9. Compute the homologies of the complex in Figure 1, or the 2-sphere, the 2-torus, etc.

Exercise 1.10. Show that every $\beta \in B_0(X)$ if and only if $\#(\beta \cap C)$ is even on every connected component C of Σ. Deduce that the dimension of $\dim(H_0(\Sigma))$ is the number of connected components of Σ.

Observation 1.11. Note that $\zeta \in Z_1(X)$ if and only if every vertex is adjacent to an even number of edges in ζ.

Exercise 1.12. Let X be connected. Define the map $\mathfrak{h} : \pi_1(X,x) \to H_1(X)$ by sending the loop γ which we may assume is a concatenation of edges $\gamma = e_1 \cdot \ldots \cdot e_n$ to the collection of edges which appear in γ odd number of times.

1. Show that $\mathfrak{h}$ is a surjective homomorphism.
2. Let $N = \left\langle \{a^2, [a,b] | \forall a, b \in \pi_1(X,x)\} \right\rangle$. Show that $\mathfrak{h}$ induces an isomorphism

$$\left(\pi_1(X)^{ab}/(\pi_1(X)^{ab})^2 \simeq \right) \quad \pi_1(X)/N \simeq H_1(X)$$

by building an inverse map or by showing $N = \ker \mathfrak{h}$. (Where else did you see relations which are products of squares and commutators?)

1.3 Cohomology

Definition 1.13 (Cohomology)**.** Now consider $C^k(\Sigma) = \operatorname{Hom}(C_k(\Sigma), \mathbf{F}) \simeq \mathbf{F}^{\Sigma^k}$, and the map $\delta : C^k \to C^{k+1}$ defined by $\delta = \partial^*$, i.e., $(\delta f)(\alpha) = f(\partial \alpha)$ for all $\alpha \in C_{k+1}$ and $f \in C^k$. The sequence (C^k, δ) is called the cochain complex of Σ, and the elements of $C^k(\Sigma)$ are called k-cochains. We define $B^k \subseteq Z^k \subseteq C^k$ by $B^k(X) = \operatorname{im} \delta$ and $Z^k = \ker \delta$ (note that $B_k \subseteq Z_k$ because again we have $\delta^2 = 0$). Finally, the *cohomology* is defined by $H^k(\Sigma) = Z^k/B^k$.

Exercise 1.14. Show that $\alpha \in Z^0$ if and only if it is constant on connected components of Σ. Deduce that $H^0(\Sigma) \simeq \mathbf{F}^{\pi_0(\Sigma)}$ where $\pi_0(\Sigma)$ are the connected components of Σ.

Observation. $\alpha \in Z^1$ if and only if for every 2-simplex σ, $\alpha = 1$ on either 0 or 2 of the edges of $\partial\sigma$.

Exercise 1.15. Assume that X is connected, and show that $H^1(X) = \mathrm{Hom}(\pi_1(X,x), \mathbf{F})$. Hint: for every $\alpha \in H^1(X)$ define the homomorphism $\phi_\alpha(\gamma)$ to be the parity of the number of times γ passes over an edge e such that $\alpha(e) = 1$.

Exercise 1.16 (Universal Coefficient Theorem)**.** Show that the natural map $H^n(X) \to \mathrm{Hom}(H_n(X), \mathbf{F})$ is a well-defined isomorphism. In particular, if they are both finite dimension, then they are isomorphic. Give an alternative proof of the previous exercise.