2 Splittings and Bass-Serre Theory

2.1 Graph of spaces

First, let us give the definition of a graph á la Serre.

Definition 2.1 (Graph à la Serre)**.** A *graph* Γ is a 5 tuple $(V, \vec{E}, o, t, \bar{\cdot})$ satisfying:

1. $V, \vec{E}$ are its sets of *vertices* and *(directed) edges* respectively;
2. $o, t : \vec{E} \to V$ are its *origin* and *terminus* maps; and
3. $\bar{\cdot} : \vec{E} \to \vec{E}$ satisfies $\bar{\bar{e}} = e$ and $\bar{e} \neq e$ for all $e \in \vec{E}$, and $o(\bar{e}) = t(e)$.

Denote by E be the collection of pairs $\{e, \bar{e}\}$ of elements $e \in \vec{E}$. Elements of E are the *(undirected) edges* of Γ.

Note that in this definition, the graph is undirected, but the collection $\vec{E}$ is the collection of directed edges. Each edge has an origin vertex $o(e)$ and a terminal vertex $t(e)$, and $\bar{e}$ stands for the same edge with the reverse direction.

Observation 2.2. Every graph gives rise to a topological graph (i.e., 1-dimensional CW complex) in an obvious way. We will not really distinguish between the two unless needed.

Now we are ready of the main definition in this course:

Definition 2.3 (A graph of spaces)**.** A *graph of spaces* (GOS) $\mathcal{X}$ is a 4 tuple

$$(\Gamma, \{X_v\}, \{X_e\}, \{i_e : X_e \to X_v\})$$

consisting of

1. a graph $\Gamma = (V, \vec{E}, o, t, \bar{\cdot})$,
2. a *vertex space* X_v (*edge space* X_e) for each vertex $v \in V$ (resp. edge $e \in \vec{E}$ so that $X_e = X_{\bar{e}}$), and
3. a continuous map $i_e : X_e \to X_{t(e)}$ for each $e \in \vec{E}$.

We will say that the graph of spaces is π_1*-injective* if all the maps i_e are π_1-injective[1].

Its *realization* is the quotient

$$X = (\coprod_v X_v) \coprod (\coprod_e X_e \times [-1,1]) / \sim$$

where $\sim$ is the equivalence relation generated by

$$X_e \times [-1,1] \ni (x,1) \sim i_e(x) \in X_{t(e)} \text{ and}$$
$$X_e \times [-1,1] \ni (x,t) \sim (x,-t) \in X_{\bar{e}} \times [-1,1].$$

for all $e \in \vec{E}$, $t \in [-1,1]$, $x \in X_e$.

We say that a space X *splits*[2] if X is the realization of a graph of spaces.

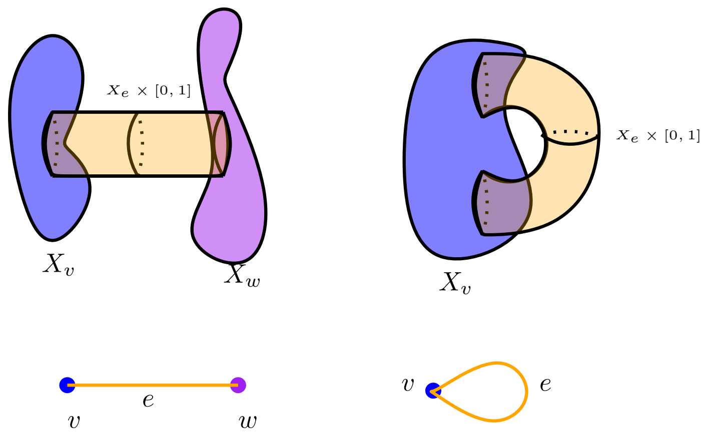

Figure 2: Graph of spaces with underlining graph an edge and a loop

See Figure 2 for two examples of a graph of spaces in which the underlining graph has only one edge.

Exercise 2.4. A *pair of pants* is a 3-holed sphere, i.e., the compact surface of genus 0 and 3 boundary components. Prove that every closed orientable surface of genus ≥ 2 splits over simple closed curves into pairs of pants. That is, every closed orientable surface is the realization of graph of groups in which all vertex spaces are pairs of pants, and all edge spaces are $\mathbb{S}^1$. Show that this is a π_1-injective splitting. How many curves must be in such a splitting?

Definition 2.5. An action of a group G on a graph Γ is an action of G on $V, \vec{E}$ such that all the maps $o, t, \bar{\ }$ are G-equivariant[3].

The action $G \curvearrowright \Gamma$ is *without inversions* if $ge \neq \bar{e}$ for all $g \in G, e \in \vec{E}$. Note that in this case Γ/G is again a graph.

Observation 2.6. The assumption that the action is without inversions is not a strong one, as if $G \curvearrowright \Gamma$ has inversions, then the action of G on the graph obtained from Γ by making each edge into two edges has no inversions.

Exercise 2.7. 1. Explain why a cover of a graph of spaces is a graph of spaces.

2. Show that there is a continuous map from the realization X of a graph of spaces $\mathcal{X}$ to (the topological model of) its underlining graph Γ such that every curve in Γ can be lifted to X.

3. Deduce that the universal cover of a graph of spaces is a tree of spaces (i.e., a graph of spaces in which the underlining graph is a tree).

4. Explain how the action of $\pi_1(X)$ on the universal cover of X by deck transformations induces an action on the underlining tree.

[1]A continuous map $f : X \to Y$ between path connected spaces is π_1-*injective* if $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

[2]Usually we will say that it splits *over* a collection of spaces to mean that the edge spaces are in this collection

[3]Given actions $G \curvearrowright X$ and $G \curvearrowright Y$ a function $f : X \to Y$ is G-equivariant if $f(gx) = gf(x)$ for all $g \in G, x \in X$

2.2 Graph of spaces with one edge

If a space X splits then it splits also over a graph with one edge. Therefore, the case of a splitting with one edge is especially important.

Assume that $\mathcal{X}$ is a π_1-injective graph of spaces whose underlining graph Γ has one edge, and let X be its realization. Then, there are two cases to consider:

Case 1. Separating. Γ has 2 vertices $\{v, w\}$, and the edge e connects them.

Definition 2.8 (Free product with amalgamation)**.** Let A, B, C be groups, and let $\phi : C \to A, \psi : C \to B$ be two monomorphisms[4]. Then the pushout $A *_C B$ is called a *free product with amalgamation of* A, B *over* C (or simply *amalgam* or *amalgamation*). It is the group

$$A *_C B = A * B \,/\, \forall c \in C,\ \phi(c) = \psi(c).$$

The amalgam is *non-trivial* is $\phi(C) \neq A$ or $\psi(C) \neq B$.

Exercise 2.9. Show that $\pi_1(X)$ is the free product with amalgamation of $\pi_1(X_v)$ and $\pi_1(X_w)$ along $\pi_1(X_e)$.

Case 2. Non-separating. In this case Γ has one vertex v and e is a loop at v.

Definition 2.10 (HNN extension)**.** Let A, C be groups, and let $\phi, \psi : C \to A$ be two monomorphisms. Then the *HNN extension of* A *over* C is the group

$$A *_C = A * \langle t \rangle \,/\, \forall c \in C,\ t\phi(c)t^{-1} = \psi(c).$$

The element t is called the *stable letter* of the HNN extension. An HNN is always *non-trivial*.

Exercise 2.11. Show that if Γ has one vertex v and one edge e, then $\pi_1(X)$ is an HNN extension of $\pi_1(X_v)$ over $\pi_1(X_e)$. What is the stable letter?

Definition 2.12. We say that a group G splits[5] if it is a non-trivial amalgam or an HNN extension.

@ In class[6] we will discuss how the structure of such groups is fairly easy to understand (normal forms, and Britton's lemma)

2.3 Graph of groups

In general, given a π_1-injective graph of spaces $\mathcal{X}$, we can associate to it a graph of groups (defined below), and the fundamental group of its realization is the fundamental group of the associated graph of groups.

Definition 2.13. A *graph of groups (GOG)* $\mathcal{G}$ is a 4-tuple $(\Gamma, \{G_v\}, \{G_e\}, \{\iota_e : G_e \to G_v\})$ consisting of a graph $\Gamma = (V, E, o, t, \bar{\cdot})$, a group G_v (resp. G_e) for each $v \in V$ (resp. edge $e \in E$ such that $G_e = G_{\bar{e}}$), and monomorphisms $\iota_e : G_e \to G_{t(e)}$.

[4] A *monomorphism* is an injective homomorphism

[5] Again, we will usually be interested in specific splitting in which we specify the possible edge and/or vertex groups

[6] The symbol @ means that we will cover something in class,

Observation 2.14. If $\mathcal{X} = (\Gamma, \{X_v\}, \{X_e\}, \{i_e : X_e \to X_v\})$ is a π_1-injective graph of spaces, then by definition $\mathcal{G} = (\Gamma, \{\pi_1(X_v)\}, \{\pi_1(X_e)\}, \{(i_e)_* : \pi_1(X_e) \to \pi_1(X_v)\})$ is a graph of groups.

When $\mathcal{X}$ is not π_1-injective one does not get a graph of groups this way, however, one can remedy this by considering instead of $\pi_1(X_v)$ and $\pi_1(X_e)$ their image in $\pi_1(X)$.

We would like to define the fundamental group of a graph of groups in such a way that would fit with the observation above. That is, if $\mathcal{G}$ is the graph of groups associated with a π_1-injective graph of spaces, then $\pi_1(\mathcal{G})$ is isomorphic to the fundamental group of the realization of $\mathcal{X}$.

There are several ways of defining the fundamental group of a graph of groups. Here are three: as a subgroup, as a quotient and from a topological space.

Definition 2.15 (the group $\mathcal{F}(\mathcal{G})$)**.** Define first the group

$$\mathcal{F}(\mathcal{G}) = *_{v\in V} G_v * F(\vec{E}) \quad / \quad \begin{array}{l} \forall e \in \vec{E},\ \bar{e} = e^{-1} \\ \forall e \in \vec{E},\ \forall g \in G_e,\ e\iota_e(g)e^{-1} = \iota_{\bar{e}}(g) \end{array}$$

Definition 2.16 (As a subgroup)**.** Now, for a vertex $* \in V$ define $\pi_1(\mathcal{G}, *)$ to be the subgroup of elements $g_1e_1g_2e_2\dots g_ne_ng_{n+1}$ in $\mathcal{F}(\mathcal{G})$ so that $e_1e_2\dots e_n$ is a closed path based at $*$, and $g_i \in G_{o(e_i)}$, and $g_{n+1} \in G_*$.

Definition 2.17 (As a quotient)**.** Choose a spanning tree $T \subseteq \Gamma$, and set $\pi_1(\mathcal{G}, T)$ to be the quotient of $\mathcal{F}(\mathcal{G})$ by the normal closure of $\{e \mid e \in \vec{E}(T)\}$.

Definition 2.18 (From topological fundamental group)**.** Given $\mathcal{G}$. Build a graph of spaces $\mathcal{X}$ whose induced graph of groups is $\mathcal{G}$ by choosing for each vertex group G_v a space (e.g. a polygonal complex or a simplicial complex) with $\pi_1(X_v) = G_v$. Similarly a polygonal complex for each edge group X_e. Let i_e be continuous maps $X_e \to X_v$ such that $(i_e)_* = \iota_e$. Define $\pi_1(\mathcal{G}, \mathcal{X})$ to be the fundamental group of the realization of $\mathcal{X}$.

Exercise 2.19. * Show all 3 definitions (and all possible choices within each) give rise to isomorphic groups.

@ What is so nice about amalgams and HNN extensions?

2.4 Bass-Serre Theory

Theorem 2.20 (From action to GOG)**.** *Let G act on a tree T without inversions, then G is the fundamental group of a graph of groups $\mathcal{G}$ whose underlining graph is T/G, its vertex groups are stabilizers of vertices, and its edge groups are stabilizers of edges, and its inclusion maps ι are inclusions of edge stabilizers in their endpoint stabilizer.*

Proof. @ □

Theorem 2.21 (From GOG to action)**.** *If G is the fundamental group of a graph of groups $\mathcal{G}$, then $G \curvearrowright T$ such that $\mathcal{G}$ is the graph of groups corresponding to this action.*

Proof. By the third definition, $G = \pi_1(\mathcal{G})$ is the fundamental group of a π_1-injective graph of spaces $\mathcal{X}$. By Exercise 2.7, its universal cover is a tree of spaces, and G acts on it. □

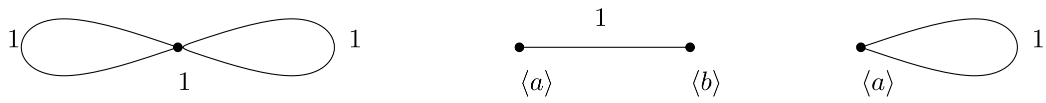

Figure 3: Three free splittings of the free group.

Exercise 2.22. Describe the trees (and the action on it) corresponding to the following splittings:

1. The splitting of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ over the simple closed curve $\{p\} \times \mathbb{S}^1$.
2. The Klein bottle $\mathbb{K}^2$ decomposes as a connected sum of two real projective planes $\mathbb{P}^2$.
3. The Baumslag Solitar group $B(1,2) = \left\langle a, t | tat^{-1} = a^2 \right\rangle$ is an HNN of the infinite cyclic group $\langle a \rangle$ with stable letter t.
4. The splitting of a closed orientable surface over simple closed curves into pairs of pants.
5. The *free* splittings, i.e., splittings with trivial edge groups, of the free group $F(a,b)$ in Figure 3.

Theorem 2.23. *A finitely generated group splits if and only if it acts on a tree without a global fixed point.*

Proof. @ □