3 Splittings of two-dimensional complexes

In the previous section we saw how one can start with a splitting of a space, and obtain a splitting of its fundamental group. We also saw how one can start with a splitting of a group G, and construct *a* space whose fundamental group is G which has a corresponding splitting.

What we often want is to find a splitting of a *given* space X from a splitting of its fundamental group. *Patterns* are exactly the tool for the job. Unfortunately we will have to pay a small price – it will not induce the same splitting of $\pi_1(X)$ but some other splitting with similar properties.

In this section we only treat the case of patterns on two-dimensional complexes. Later, we will see how to use similar ideas to tackle 3-manifolds.

3.1 Patterns and tracks

Assume throughout this section that X is a 2-dimensional simplicial complex.

Definition 3.1. A *pattern* in X is an embedded graph $A \subset X$ such that:

- the vertices of A lie in the interior of edges of A,
- the interior of edges of A are straight lines in the interior of 2-simplices of X,
- for every 2-simplex σ of X and every vertex v of A on $\partial\sigma$, there exists a unique edge of A in σ which is incident to v.

A *track* is a connected (component of a) pattern.

Example 3.2.

1. Let X be a graph, then a finite collection of points A in the interior of its edges is a pattern.
2. If X is a triangulated surface, then any finite disjoint collection of simple closed curves (and proper arcs) on X which avoid the vertices of the triangulation is (isotopic) a pattern.
3. More generally a pattern looks something like Figure 4. Note that by condition 3, the each track of the pattern must continue into every 2-simplex it meets.

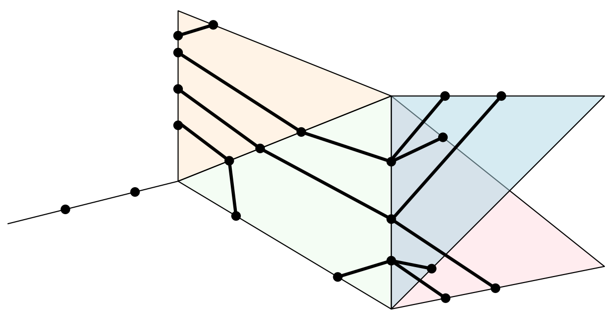

Figure 4: An example of a straight pattern on a 2-dimensional simplicial complex.

Definition 3.3. Let Σ, Σ' be (abstract) simplicial complexes. A *simplicial map* is a map $f : \Sigma^0 \to \Sigma'^0$ such that if $\sigma \in \Sigma$ then $f(\sigma) \in \Sigma'$.

Note that a simplicial map might send a simplex to a simplex of lower dimension. Also note that a simplicial map induces a continuous map between the realizations $|\Sigma| \to |\Sigma'|$. As usual we will say that the map f between the realizations is simplicial if it is induced from a simplicial map.

Observation 3.4. If X, Y are 2-dimensional simplicial complexes, A is a pattern on X, and $f : Y \to X$ is a simplicial map then the preimage $B = f^{-1}A$ is pattern on Y.

Exercise 3.5. Let A be a pattern on X, then we define $\alpha_A : X^1 \to \mathbb{Z}_{\geq 0}$ by $\alpha_A(e) = |A \cap e|$ for every edge $e \in X^1$.

1. Show that for a function $\alpha : X^1 \to \mathbb{Z}_{\geq 0}$ the following are equivalent:
   - $\alpha = \alpha_A$ for a (unique) pattern A.
   - the function α satisfies that for every 2-simplex σ in X with sides e_1, e_2, e_3 there exists $m_\sigma \in \mathbb{Z}_{\geq 0}$ for which $f(e_1)+f(e_2)+f(e_3) = 2m_\sigma$ and $f(e_i) \leq m_\sigma$ for $i = 1, 2, 3$.

   Deduce that $\alpha_A \mod 2$ is a 1-cocyle, i.e. in $Z^1(X)$.
2. Show that a track τ separates X if and only if $\alpha_\tau \in B^1(X)$.

Definition 3.6. We can now define *addition* $A + B$ of the patterns A, B to be the unique pattern which satisfies $\alpha_{A+B} = \alpha_A + \alpha_B$.

@ Examples

3.2 Fiber bundles

Definition 3.7 (Fiber bundle)**.** Let F, E, B be topological spaces, we say that $f : E \to B$ *is an F-bundle over B*, if for every $x \in B$ there exists an open neighborhood $x \in U \subseteq B$ such that $V = f^{-1}(U)$ is homeomorphic to $F \times U$ such that the map $f|_V$ is the projection on the second coordinate. In this case B is called the *base*, and F is called the *fiber*.

Example 3.8. The projection $F \times B \to B$ is an F-bundle over B. Such a bundle is called *trivial*.

Exercise 3.9.
1. Show that a cover of X of degree $k \in \mathbb{N} \cup \{\infty\}$ is a $\{1, \ldots, k\}$-bundle over X.
2. Show that there are exactly two I-bundles (interval bundles) over $\mathbb{S}^1$: the trivial bundle (i.e, the annulus $\mathbb{S}^1 \times I$), and the Möbius band. In general I-bundles are characterized by $H^1(B)$, why?
3. Explain why a pattern A has a neighborhood $\mathcal{N}(A)$ which avoids the vertices of X and is a $(-1, 1)$-bundle over A such that A corresponds to $\{0\} \times A$.

Remark 3.10. Exercise 3.9.2 can be generalized to show that the F-bundles over $\mathbb{S}^1$ are classified by conjugacy classes of the elements of $\text{Homeo}(F)/\text{Homeo}_0(F)$ where $\text{Homeo}_0(F)$ is the isotopy group, i.e., the identity component of the topological group of self-homeomorphisms $\text{Homeo}(F)$. In particular, there are two $\mathbb{D}^n$-bundles over $\mathbb{S}^1$.

Definition 3.11 (tubular neighborhood of pattern)**.** We will refer to such a neighborhood $\mathcal{N}(A)$ (as in Exercise 3.9.3) as a *tubular* neighborhood of A.

Remark 3.12. One can define patterns for more general topological spaces. A *pattern* in a topological space X is a closed subset $A \subset X$ which has an open neighborhood U in X which is a $(-1,1)$-bundle over A such that A corresponds to $\{0\} \times A$. With this definition one can prove similar results, however this will not be needed for our purposes

Definition 3.13. A track τ is *one-sided* (resp. *two-sided*) if $\mathcal{N}(\tau) - \tau$ is connected (resp. has two components). We will call a pattern *2-sided* if all of its tracks are 2-sided.

Exercise 3.14. Show that $\mathcal{N}(\tau) - \tau$ has at most 2 connected components, and

- if τ is 1-sided, then $\pi_1(\mathcal{N}(\tau) - \tau)$ has index 2 in $\pi_1(\mathcal{N}(\tau)) = \pi_1(\tau)$.
- if τ is 2-sided then $\mathcal{N}(\tau) = \tau \times (-1,1)$ is the trivial bundle.

Exercise 3.15. If A is a 2-sided pattern on X then X splits over the tracks of the pattern.

Exercise 3.16. If $\tau_1, \ldots, \tau_n$ are disjoint one-sided tracks, then the corresponding cocycles $\alpha_1, \ldots, \alpha_n$ are independent vectors in $H^1(X)$.

Remark 3.17. By tweaking the definition of a graph to allow $\bar{e} = e$. One can define a dual graph also for patterns which are not 2-sided. However, this is not really necessary, as whenever we have a 1-sided tracks, the boundary of a small tubular neighborhood around it 2-sided.

3.3 @ From actions on trees to patterns

Let G be the fundamental group of a simplicial complex X. Let $G \curvearrowright \widetilde{X}$ be the action of G on the universal cover of X by deck transformations. Let T be a tree, and let $G \curvearrowright T$ be an action without inversions.

We build a G-equivariant map $\widetilde{X} \to T$ by defining it on vertices, edges and 2-simplices. This map will (most likely) not be simplicial but will be piecewise linear – i.e., it will be simplicial once we subdivide $\widetilde{X}$ enough. Don't worry, we will come back to this notion later on, when we talk about 3-manifolds.

<u>Vertices:</u> Choose orbit representatives $x_1, \ldots, x_n$ for the action of G on the vertices of $\widetilde{X}$, and arbitrary vertices $v_1, \ldots, v_n \in T$. Now, define the map f on $x_1, \ldots, x_n$ by $f(x_i) = v_i$. Extend it to all vertices G-equivariantly by $f(gx_i) = gv_i$. This is well-defined because the action $G \curvearrowright \widetilde{X}$ is free.

<u>Edges:</u> Now for each edge $e \subset \widetilde{X}$, let f map e linearly to the geodesic connecting $f(o(e))$ and $f(t(e))$.

<u>2-Simplices:</u> For each 2-simplex σ, $f(\partial\sigma)$ is a tripod (see Figure 5), and one can extend the map f to σ by dividing it into 4 simplices and mapping them linearly so that the middle is sent to the vertex at the "middle" of the tripod.

If M is the pattern on T which is the collection of midpoints of edges of T (shown in dark green in Figure 5), then $\widetilde{A}_f = f^{-1}(M)$ is a pattern on $\widetilde{X}$ which is G invariant. Hence it defines a pattern A_f on X. This pattern is 2-sided because the action is without inversions, and so we get a splitting of X. Moreover, there is a G-equivariant map $F : T_{\widetilde{A}_f} \to T$ where $T_{\widetilde{A}_f}$ is the tree dual to $\widetilde{A}_f$.

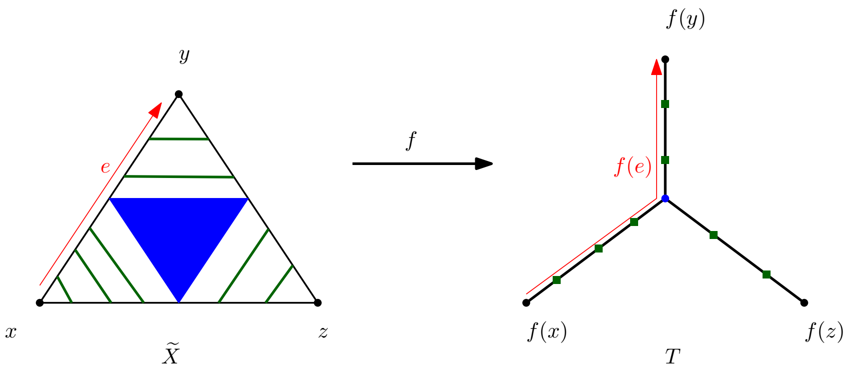

Figure 5: The map $f : \widetilde{X} \to T$ and the pattern $\widetilde{A}_f$.

3.4 @ Bounding the number of tracks

Definition 3.18. Two tracks τ_-, τ_+ are parallel, if there exists a 2-sided track τ and a neighborhood $\mathcal{N}(\tau) = \tau \times (-1,1)$ such that $\tau \times \{\pm 1/2\} = \tau_\pm$.

In other words, the two tracks are 2-sided and the region between them is a product that does not pass through vertices.

Theorem 3.19 (Kneser's Bound)**.** *If X is a finite 2-dimensional simplicial complex then there exists $\kappa(X) \geq 0$, such that if A is a straight pattern on X with more than $\kappa(X)$ tracks then A contains two parallel tracks.*

Proof. @ □

We summarize the discussion above in the following theorem

Theorem 3.20. *Let $G = \pi_1(X)$ then there exists $\kappa(X)$, such that for every action $G \curvearrowright T$ without inversions, there exists a tree T' which is dual to some pattern in $\widetilde{X}$ and a G-equivariant PL map $T' \to T$, so that T'/G has at most $\kappa(X)$ edges. Moreover, the edge and vertex stabilizers of $G \curvearrowright T'$ are finitely generated.*

@ Why do we care about such a theorem?