6 Heegaard Splitting

We will first show that *every (!!!)* compact 3-manifold can be decomposed into two handlebodies (see below) along a surface. This will have immediate consequences on $\pi_1(M)$, but is not at all canonical like the splittings that will follow.

Definition 6.1. A *handlebody* H is a 3-manifold (with boundary) that has a splitting with one vertex space which is a 3-ball $\mathbb{D}^3$, and whose edge spaces are proper disks $\mathbb{D}^2$ so that the attaching maps are embeddings $\mathbb{D}^2 \hookrightarrow \partial\mathbb{D}^3$ with disjoint image. The number of edges is the *number of handles* (or the *genus* of the handlebody). See Figure 7.

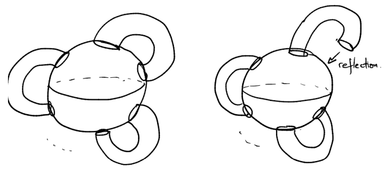

Figure 7: The oriented and unoriented handlebodies

Remark 6.2. Note that the genus of the handlebody is the genus $g(\partial H)$ of ∂H, where the genus of a closed (orientable or non-orientable) surface F is defined by $g(F) = 1 - \frac{1}{2}\chi(F)$.

Exercise 6.3. Show that up to homeomorphism a handlebody is determined by its genus and orientation.

Exercise 6.4. Show that if S is a surface with boundary, then $S \times I$ is a handlebody, and $S \times I$ is orientable if and only if S is.

Definition 6.5. A *Heegaard splitting* of a closed 3-manifold, is a splitting along a surface S into two handlebodies. In other words, there are two handlebodies H_1, H_2 in M embedded in M so that $M = H_1 \cup H_2$ and $\partial H_1 = \partial H_2 = H_1 \cap H_2$.

Exercise 6.6. Show that every 3-manifold (with boundary) which has a splitting along proper disks into 3-balls is a handlebody.

Exercise 6.7. Show that every 3-manifold has a Heegaard splitting. Moreover, show that the manifold is orientable if and only if each of the handlebodies is. Hint: Let Σ be a triangulation of M, consider the neighborhoods of the 1-skeleton of Σ, and the 1-skeleton of $\Sigma^\perp$.

Exercise 6.8. Find a Heegaard splitting of $\mathbb{S}^3$ for each genus g.
Fing a Heegaard splitting of $\mathbb{P}^3$.

@ Heegaard diagram

Exercise 6.9. Assume M is a closed (say, orientable) 3-manifold, and $M = H_1 \cup H_2$ is a Heegaard splitting and each of H_i has n-handles. Show that its fundamental group has a balanced presentation, i.e., a presentation $\pi_1(M) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ where the number of generators equals the number of relations.

The last exercise shows for example that $\mathbb{Z}^n$ for $n \geq 4$ and $(C_2)^m$ for $m \geq 2$ cannot be the fundamental group of a closed 3-manifold. (Compare this to the fact that every finitely presented group is the fundamental of a closed 4-manifold)

The Heegaard splitting has two major flaws: it is not canonical, and it is not π_1-injective.

@ Alexander's and Waldhausen's Theorem.

@ Reidemeister-Singer Theorem.

@ Poincaré Conjecture.

6.1 Lens space

Definition 6.10. A lens space is a 3-manifold with a Heegaard splitting over the torus.

Exercise 6.11. For every p, q such that $\gcd(p, q) = 1$ The space $L(p, q)$ is obtained by taking the quotient of $\mathbb{S}^3 \subset \mathbb{C}^2$ by the cyclic group of order p generated by the isometry $(x, y) \mapsto (e^{2\pi i/p}x, e^{2\pi iq/p}y)$. Show that the space $L(p, q)$ is a lens space, and that its fundamental group is cyclic of order p.

@ What is the Heegaard diagram of $L(p, q)$?