5 The basics of 3-manifolds

5.1 Piecewise-Linear structure

Definition 5.1. A *topological n-manifold* (with boundary) is a Hausdorff second-countable space M which is locally homeomorphic to open subsets of $\mathbb{R}^{n-1} \times [0, \infty)$.

The *boundary* ∂M of M are those points in M which are sent to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^{n-1} \times [0, \infty)$ under the local homeomorphisms. A manifold is *closed* if it is compact and $\partial M = \varnothing$.

Observation 5.2. The boundary ∂M is an $(n-1)$-manifold without boundary (i.e., $\partial(\partial M) = \varnothing$).

Example 5.3. Some basic examples and constructions of 3-manifolds:

- The *3-sphere* $\mathbb{S}^3 = \{v \in \mathbb{R}^4 | \ \|v\| = 1\}$; the *3-ball* $\mathbb{D}^3 = \{v \in \mathbb{R}^3 | \ \|v\| \leq 1\}$; the *3-torus* $\mathbb{T}^3 = (\mathbb{S}^1)^3$.
- The real projective 3-space:

$$\begin{aligned} \mathbb{P}^3 &= (\mathbb{R}^4 - 0) \ / \ \forall \lambda \neq 0, \ x \sim \lambda x \\ &= \mathbb{S}^3 \ / \ x \sim -x \\ &= \mathbb{D}^3 \ / \ \forall x \in \partial \mathbb{D}^3, \ x \sim -x. \end{aligned}$$

- Products $F \times \mathbb{S}^1$ or $F \times [0,1]$ where F is a surface. For example, the *full torus* $\mathbb{D}^2 \times \mathbb{S}^1$ is a 3-manifold with torus boundary.
- More generally, surface-bundles over $\mathbb{S}^1$. As we saw, such a bundle can be described as the *mapping torus* of a homeomorphism $\phi : F \to F$ of a surface F, i.e. as the space

$$M_\phi = F \times [0,1] \ / \ \forall x \in F, \ (x,1) \sim (\phi(x), 0).$$

Manifolds are in general quite wild objects, but it turns out that in dimension ≤ 3, they are more tame. In particular, they have a unique smooth structure (@ in class) and a unique piecewise linear structure (defined below).

Definition 5.4. Let Σ be a simplicial complex. A *subdivision* of Σ is a simplicial complex Σ' such that $|\Sigma| \cong |\Sigma'|$ and each simplex of Σ' is contained in a simplex of Σ. See Figure 6.

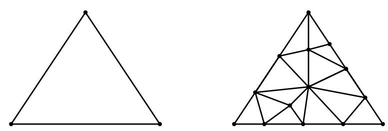

Figure 6: The simplex on the left is subdivided on the right

A map $f : |\Sigma| \to |\Lambda|$ is *piecewise linear* (PL) if there exist subdivisions Σ', Λ' of Σ, Λ such that f is the realization of a simplicial map $\Sigma' \to \Lambda'$.

A *triangulation of a space* X is a pair (Σ, h) of a simplicial complex Σ and a homeomorphism $h : |\Sigma| \to X$. Two triangulations $(\Sigma_1, h_1), (\Sigma_2, h_2)$ are *compatible* if $h_2^{-1} h_1$ is PL.

A *PL structure* on a **3-manifold** is a (compatibility class of) triangulations.

Fact 5.5. There is a unique PL structure on a 3-manifold up to PL homeomorphism.

@ The actual definition of a PL structure for n-manifolds will be discussed in class.

Definition 5.6. A *PL submanifold* N of a PL manifold M, is a subcomplex Λ of a triangulation (Σ, h) of M such that $(\Lambda, h|_{|\Lambda|})$ is a triangulation of the manifold N.

A submanifold is *proper* is $\partial N = \partial M \cap N$.

Fact 5.7. Every proper PL compact m-submanifold N of a 3-manifold M has a *tubular* neighborhood. I.e., an open neighborhood which is an $\mathbb{R}^{3-m}$-bundle over N.

@ Why PL submanifolds and not to subspaces which are manifolds?

Example 5.8 (Knots, links and their complements)**.** A PL submanifold $K \cong \mathbb{S}^1$ (more generally, $L \cong \mathbb{S}^1 \coprod \ldots \coprod \mathbb{S}^1$) in $\mathbb{S}^3$ is called a *knot* (resp. a *link*). Every knot (resp. ink) has a tubular neighborhood $\mathcal{N}(K)$ which is a full torus (resp. a disjoint collection of full tori). Removing it from $\mathbb{S}^3$ we get one of the most important sources for 3-manifolds with torus boundaries: *knot (and link) complements.*

Example 5.9 (Dehn surgery)**.** Let L be a link in $\mathbb{S}^3$. Consider the link complement $M = \mathbb{S}^3 - \mathcal{N}(L)$, and to each torus boundary component glue back a solid torus but using a different identification of its boundary. This is called Dehn surgery. It is a theorem of Lickorish-Wallace that every closed 3-manifold can be obtained in this way!

5.2 Orientation

Definition 5.10. We will say that two ordering of the vertices of a simplex are equivalent if they differ by an even permutation, that is $(v_0, \ldots, v_n) \sim (v_{\pi(0)}, \ldots, v_{\pi(n)})$ for every $\pi \in A_{n+1}$ (the alternating subgroup of S_{n+1}). An *oriented* n-simplex $[v_0, \ldots, v_n]$ is one of the two equivalence classes of ordering of the vertices up to the alternating group. The other orientation will be denoted by $-[v_0, \ldots, v_n]$. So that $[v_0, \ldots, v_n] = \mathrm{sgn}(\pi)[v_{\pi(0)}, \ldots, v_{\pi(n)}]$ for all $\pi \in S_n$.

Let $\sigma = [v_0, \ldots, v_n]$ be an oriented simplex. For every $(n-1)$-face τ of σ the orientation of τ induced by σ is $(-1)^i[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$.

We say that a PL n-manifold M is *oriented* if there is an orientation on each simplex such that for all $(n-1)$-simplex τ the orientation induced by each of its incident n-simplexes are opposite.

Remark 5.11. Now that we have defined orientation of simplices, we can actually define the simplicial homology with $\mathbb{Z}$ coefficients. The chains are the $\mathbb{Z}$-span of oriented simplices, where we mod out the relation $(-1) \cdot \sigma = -\sigma$, i.e., the simplex with the reverse orientation. The boundary map ∂ is defined on an oriented simplex σ to be $\partial\sigma = \sum \tau$ where the sum ranges over the faces τ of σ with the induced orientation.

Note, that this definition shows that a closed manifold has an orientation if and only if $H_n(M; \mathbb{Z}) \simeq \mathbb{Z}$. Therefore, being orientable is actually a property of the manifold and not the specific triangulation.

Exercise 5.12. Show that every non-orientable PL n-manifold M has a double cover which is orientable.

Exercise 5.13.

- Is $\mathbb{P}^3$ orientable?

- Is every PL submanifold of an oriented (PL) 3-manifold oriented?

Exercise 5.14. Show that the boundary of an orientable manifold is orientable. What about non-orientable manifolds?

Recall that an $\mathbb{R}^2$-bundle over a $\mathbb{S}^1$ are characterized by the mapping class group of $\mathbb{R}^2$. The mapping class group of $\mathbb{R}^2$ has two elements the orientation preserving and reversing classes. So a $\mathbb{R}^2$-bundle over $\mathbb{S}^1$ is either trivial or twisted.

Exercise 5.15. The manifold M is orientable if the tubular neighborhood of any simple closed curve is a trivial bundle.

@ What is the relation between the orientability of a manifold M, the orientability of a submanifold N and its tubular neighborhood?

5.3 Poincaré Duality

Theorem 5.16 (Poincaré Duality)**.** *Let M be closed connected n-manifold. Then for all k, $H^{n-k}(M) \simeq H_k(M)$.*

"Proof". Let us show for a 3-manifold that $H_1(M) \simeq H^2(M)$. Consider a triangulation Σ of M. First, construct the dual complex $\Sigma^{\perp}$ as follows: Place a vertex in the interior of each 3-cell, connect two such vertices by an edge if the 3-cells share a 2-cell face. Next, around every edge of Σ there are some number of 3-cells cyclically connected, in the dual graph $(\Sigma^{\perp})^{(1)}$ we have built so far there is a corresponding circle, attach the boundary of a 2-disk to this circle. Finally, around every vertex we have constructed a polygonal complex of a sphere $\mathbb{S}^2$, fill the sphere with a 3-ball.

Instead of the homologies $H_k(M) = H_k(\Sigma)$ for the original simplicial complex, one can consider the homologies $H_k(\Sigma^{\perp})$ of the dual complex. Even though it is not a simplicial complex anymore, the definition is basically the same, and one has $H_k(M) = H_k(\Sigma^{\perp}) = H_k(\Sigma)$. So it suffices to show that $H_1(\Sigma^{\perp}) \simeq H^2(\Sigma)$:

Exercise 5.17. Consider the map $c \in H_1(\Sigma^{\perp}) \mapsto \alpha_c \in H^2(\Sigma)$ defined by $\alpha_c(\sigma) = |\sigma \cap c|$ for every $\sigma \in \Sigma^2$. Show that it is a well-defined isomorphism.

□

Those who are familiar with the Poincaré Duality Theorem, might find that the above statement is missing the assumption of *orientability*. The reason that this assumption is not needed, is that the (co)homologies are defined over $\mathbb{F}_2$. With a little more care, the "proof" above can be turned into a proof of the usual formulation of the Poincaré Duality Theorem:

Theorem 5.18 (Poincaré Duality)**.** *Let M be a closed connected* orientable *n-manifold. Then for all k, $H^{n-k}(M;\mathbb{Z}) \simeq H_k(M;\mathbb{Z})$.*