7 Connected sums and prime decomposition

Definition 7.1. Let M_1, M_2 be two 3-manifolds (possibly with boundary). A manifold M is a connected sum of M_1 and M_2, denoted $M_1 \# M_2$, if there exists embedded closed balls $B_i \subset \text{int}(M_i)$ and embeddings $h_i : M_i - \text{int}(B_i) \to M$ such that $\text{im}\, h_1 \cap \text{im}\, h_2 = h_1(\partial B_1) = h_2(\partial B_2)$. In other words, M is obtained by gluing $M_1 - \text{int}(B_1)$ to $M_2 - \text{int}(B_2)$ along ∂B_i. If M_1, M_2 are oriented, we require the connected sum to have the orientation of M_1 and M_2. I.e. the gluing of ∂B_1 to ∂B_2 reverses the induced orientation.

Fact 7.2. The connected sum is not well-defined. However the only disambiguity is in the 'orientation' of the gluing of ∂B_1 to ∂B_2 (and not in any of the other choice). If M_1, M_2 are oriented, then $M_1 \# M_2$ is well-defined, and is associative and commutative.

Definition 7.3. A 3-manifold M is *prime* if $M \ncong \mathbb{S}^3$ and whenever $M = M_1 \# M_2$ then $M_1 = \mathbb{S}^3$ or $M_2 = \mathbb{S}^3$.

Observation 7.4. If $M = M_1 \# M_2$ then $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$. Under the assumption of the Poincaré Conjecture, if M is not prime then $\pi_1(M)$ has a non-trivial as a free product.

Exercise 7.5. Use the observation above and Alexander's Theorem to show that:

- $\mathbb{D}^3$ is prime.
- $\mathbb{S}^2$–bundles over $\mathbb{S}^1$ are prime.
- The Lens spaces $L(p, q)$ are prime.

[Hint: if they are not, $M = (M_1 - B_1) \cup (M_2 - B_2)$ glued along a sphere, by the observation one of them must be simply connected. Without loss of generality, $\pi_1(M_1 - B_1) = 1$. Lift $M_1 - B_1$ to the universal cover and deduce that it is a standard 3-ball.]

Our goal is the following decomposition theorem:

Theorem 7.6 (Prime decomposition - existence)**.** *Let M be a compact 3-manifold, then there exists a decomposition $M = M_1 \# \cdots \# M_n$ such that each M_i is prime.*

Assuming the Poincaré conjecture (which is now a theorem), the prime decomposition theorem is a consequence of Grushko's decomposition. That said, we will give a proof that does not depend on the Poincaré conjecture.

It would be useful to not worry about sphere boundary components, as those are spheres which are not 2-sided. For this purpose, we define define the capping of a manifold.

Definition 7.7. Let M be a compact 3-manifold. Denote by $\hat{M}$ the manifold obtained by gluing a 3-ball to each sphere boundary component.

Exercise 7.8. If M is a compact manifold, and $\hat{M} = M_1 \# \dots \# M_n$ is a prime decomposition of $\hat{M}$, then $M = M_1 \# \dots \# M_n \# B_1 \# \dots B_k$ is a prime decomposition for M such that k is the number of sphere boundary components and B_i are 3-balls.

Definition 7.9. A 3-manifold M is *irreducible* if $M \ncong \mathbb{S}^3$ and every embedded 2-sphere bounds a 3-ball.

Exercise 7.10. • Suppose M is a 3-manifold such that $\text{int}(M)$ contains a sphere S such that $M - S$ is connected. Then $M = M_1 \# M_2$ where M_1 is a $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. [Hint: consider a small neighborhood of S together with an arc connecting its two sides.]

- Show that every irreducible manifold is prime, and deduce that the only non-irreducible prime manifolds are $\mathbb{S}^2$-bundles over $\mathbb{S}^1$.

Proof of Theorem 7.6. @ □

Exercise 7.11. If $M = M_1\#(\mathbb{S}^2 \times \mathbb{S}^1)$ and M_1 is non-orientable then $M = M_1\#P$ where P is the non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. [Hint: use the same hint as in the previous exercise to show $M = M_1'\#P$. Show that $\hat{M}_1 = \hat{M}_1' = (M - \mathcal{N}(S))\hat{}$.]

Definition 7.12. A prime decomposition $M = M_1\#\dots\#M_n$ is *normal* if either M is orientable or M is non-orientable and none of the M_i is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1 \implies$.

Theorem 7.13 (Prime decomposition - uniqueness)**.** *Let $M = M_1\#\dots\#M_n = M_1'\#\dots\#M_{n'}'$ are two normal prime decompositions of M, then $n = n'$ and up to reordering $M_i \cong M_i'$.*

Proof. @ □