8 Dehn's Lemma, the Loop and Sphere Theorems

The following 3 theorems are perhaps among the most important basic theorems in 3-manifold topology. What they all have in common is that we assume the existence of some ("non-trivial") map f from a disk or a sphere into M and we want to turn it into an embedding g (with similar "non-triviality" assumption). The existence of such an embedding is what turns many (algebraic/homotopic) properties of 3-manifolds into combinatorial/topological ones.

Theorem 8.1 (Dehn's Lemma, Papakyriakopoulos)**.** *Let M be a 3-manifold, and let $f : \mathbb{D}^2 \to M$ be a map which is injective on a small neighborhood U of $\partial\mathbb{D}^2$ and such that $f^{-1}f(U) = U$, then there exists an embedding $g : \mathbb{D}^2 \to M$ such that $g|_{\partial\mathbb{D}^2} = f|_{\partial\mathbb{D}^2}$.*

Theorem 8.2 (The Loop Theorem)**.** *Let M be a 3-manifold and $F \subset \partial M$ a connected sub-surface. If $\pi_1(F) \to \pi_1(M)$ is not injective then, there exists an embedding $g : \mathbb{D}^2 \hookrightarrow M$ such that $g|_{\partial\mathbb{D}^2}$ is a non-trivial element in $\pi_1(F)$.*

Theorem 8.3 (The Sphere Theorem)**.** *Let M be an orientable 3-manifold, if $f : \mathbb{S}^2 \to M$ is continuous and cannot be extended continuously to $\mathbb{D}^3 \to M$ then there exists an embedding $g : \mathbb{S}^2 \hookrightarrow M$ with the same property. In particular, M is reducible.*

@ $\pi_n(X)$, and stronger formulations of the theorems.

Exercise 8.4. Deduce Dehn's lemma from the loop theorem. [Hint: remove a small regular neighborhood of $f(\partial\mathbb{D}^2)$ and apply the loop theorem.]

Exercise 8.5. Deduce the following: Let $F \subset M$ be a 2-sided embedded surface, then if F is not π_1-injective, then there exists an essential simple closed curve in F (i.e., a simple closed curve that does not bound a disk in F) which bounds a disk D in M such that $D \cap F = \partial D$. [Hint: Let $f : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, F)$ be in general position to F, show that $f^{-1}(F)$ is a collection of simple closed curves, take an innermost curve $c \subset f^{-1}(F)$, show that either $f(c)$ is inessential in F in which case the number of components of $f^{-1}(F)$ can be reduced or using the loop theorem one gets the desired curve.]

For the proof of the loop theorem the main idea is to start with a map $f : \mathbb{D}^2 \to M$ which is in general position (see below). Such a map will have singularities - i.e., points where its image intersects itself. The idea is to try to replace f by a new map which has less singularities until all are gone and we get an embedding. This is done using two tools: 'cut and paste techniques', and a 'tower construction'.

8.1 Singularities of maps

Let F be a compact surface, M a 3-manifold. Let $f : (F, \partial F) \to (M, \partial M)$ be a PL map. Define $S(f) \subset F$ to be the *closure* of points $x \in F$ such that $\#(f^{-1}f(x)) > 1$. Define

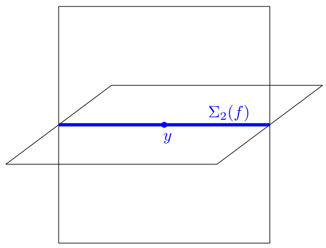
$$S_i(f) = \{x \in S(f) \mid \#(f^{-1}(f(x))) = i\}, \text{ and } \Sigma_i(f) = f(S_i(f)).$$

Note that $S_1(f)$ is not necessarily empty.

A map $f : F \to M$ is in *general position* if:

1. $S_1(f)$ and $S_3(f)$ are finite.

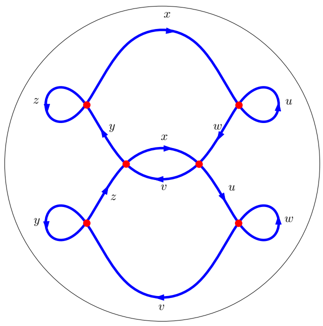
2. for all $i \geq 4$, $S_i(f) = \emptyset$.
3. $f|_{F - S_1(F)}$ is an *immersion*, i.e. it is a local embedding.
4. Locally, around every $y \in \Sigma_2(f)$ the set $\Sigma_2(f)$ is the arc of intersection of two sheets of $f(F)$ intersecting transversely. E.g. if $y \in \Sigma_2(f) \cap \mathrm{int}(M)$ then it is as in the following figure:



Exercise 8.6. Show that up to a small homotopy, every PL map f can be made into a PL map in general position.

@ Simple double curves, and how to resolve them.

Exercise 8.7. Show that the the following diagram describes the singularities for some map $f : \mathbb{D}^2 \to M$ in general position. The bold lines are the points in $S_2(f)$, arcs with same label are sent by f to the same arc in M (in the same orientation as indiacted).



8.2 Tower construction

The basic idea is that instead of resolving singularities in M it might be easier to resolve singularities in a cover. To help up produce such cover we need the following lemma.

Lemma 8.8. *Let M be a compact 3-manifold with non-empty boundary. If some component of ∂M is not a sphere, then M has a (connected) double cover.*

Exercise 8.9. Prove the lemma. [Hint: enough to show that $H_1(M) \simeq H^1(M) \neq 0$. Since there is a component $F \subset \partial M$ which is not a sphere, $H_1(F) \neq 0$. Take a non-trivial element $c \in H_1(F)$, either c is non-trivial in $H_1(M)$ and we are done, or it is the boundary of some element f in $C_2(M)$. Use f and Poincaré Duality to show that then $H^1(M) \neq 0$ (Hint: you can double M to make it closed)]

@ Proof of the Loop Theorem