V Equivalent definitions of hyperbolic spaces.

1. Thin triangles

Def: let X be a metric sp. let $x_1, x_2, x_3 \in X$. The triangle inequality guarantees that $\exists$ a tripod $\hat{x}_1$ — o — $\hat{x}_2, \hat{x}_3$ such that $d(x_i, x_j) = d(\hat{x}_i, \hat{x}_j)$.

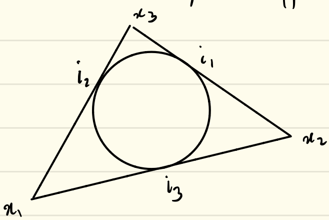
We will call it the comparison tripod. $T(\hat{x}_1, \hat{x}_2, \hat{x}_3)$

If X is geodesic, we can define a map $\Delta(x_1, x_2, x_3) \xrightarrow{t} T(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ by sending each geodesic to the corr. geodesic in T.

A triangle Δ is called δ-thin if $\forall p \in T$ $\operatorname{diam}(t^{-1}(p)) \leq \delta$.

The insize of Δ $\operatorname{insize}(\Delta)$ is $\operatorname{diam}(t^{-1}(o))$.

The pts of $t^{-1}(o)$ are called the internal pts of Δ, and denoted i_1, i_2, i_3 according to the vertex they are opposite to.



Thm Let X be a geodesic m.s. TFAE:

1. $\exists \delta_1$ such that all triangles are δ_1-slim.
2. $\exists \delta_2$ —"— δ_2-thin
3. $\exists \delta_3$ —"— have $\operatorname{insize}(\Delta) \leq \delta_3$

Pf: $2 \Rightarrow 1$ is obvious.

$1 \Rightarrow 3$: Let $\Delta(x_1, x_2, x_3)$ be a geod triang. by hypothesis it is δ_1-slim. let i_1, i_2, i_3 be its internal pts. i_1 is δ_1 close to some pt p on $[x_1, x_2]$ or $[x_1, x_3]$. say $p \in [x_1, x_2]$.

By the triangle inequality $|d(x_1, p) - d(x_1, i_1)| \leq \delta_1$.

Since $d(x_2, i_1) = d(x_2, i_3)$ it follows that $|d(x_1, p) - d(x_1, i_3)| \leq \delta_1$

and $d(p, i_3) \leq \delta_1$ (since they are on the same geod)

$\Rightarrow d(i_1, i_3) \leq 2\delta_1$.

Similarly $d(i_2, \{i_1, i_3\}) \le 2\delta_1$ and so $\operatorname{diam}\{i_1, i_2, i_3\} \le 4\delta_1$.

$3 \Rightarrow 2$: Let $\Delta(x_1, x_2, x_3)$ be a geod Δ.

Let y_2, y_3 be the 2 preimages of a pt $p \in T(x_1, x_2, x_3)$ (WLOG $p \in [\hat{x}_1, o]$ $p \neq o$.). WTS $d(y_1, y_2) \le \delta_2$ for some δ_2.

Let $x_3' \in [y_3, x_3]$ be such that $d(x_3', x_2) = d(x_3', x_1) - d(y_2, x_1)$

Therefore, y_2, y_3 are the internal pts of $\Delta(x_1, x_2, x_3')$ and it follows by hypothesis that $d(y_2, y_3) \le \delta_3$. $\square$

Ex: Show that these are equivalent to:

$\exists \delta_4$ such that all triangles $\Delta(x_1, x_2, x_3)$ satisfy

$$\inf\{\operatorname{diam}\{p_1, p_2, p_3\} \mid p_i \in [x_{i+1(3)}, x_{i+2(3)}]\} \le \delta_4.$$

2. The Gromov product

Def: Let X be a metric sp. let $x, y, w \in X$

we define $(x.y)_w = \frac{1}{2}\left(d(x,w) + d(y,w) - d(x,y)\right)$.

or equivalently $d(\hat{w}, o)$ in $T(x, y, w)$.

Note that $d(w, [x,y]) \le (x.y)_w$ + insize Δ. If Δ is δ-thin then

$$|d(w, [x,y]) - (x.y)_w| \le \delta.$$

Def: Let X be a metric sp. we say that X is (δ)-hyperbolic if $\forall x, y, z, w$ we have:

$$(x.y)_w \ge \min\{(x.z)_w, (y.z)_w\} - \delta. \quad \circledast$$

Note that this does not assume that X is geodesic.

Ex: If $\exists w$ s.t $\circledast$ holds $\forall x, y, z$ then it holds $\forall x, y, z, w$ replacing δ by 2δ.

Expanding the expressions in $\circledast$ we get

$$d(x,y) + d(z,w) \le \max\{d(x,z) + d(y,w), d(y,z) + d(x,w)\} + 2\delta.$$

In other words, we look at the "tetrahedron" x, y, z, w.

and we sum the lengths of opposite sides.

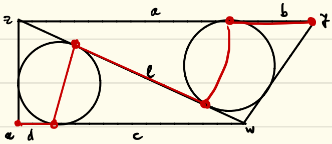
WLOG $S := d(x,z) + d(y,w) \le M := d(y,z) + d(x,w) \le L := d(x,y) + d(z,w)$

Then $\circledast$ tells us that $L \le M + 2\delta$.

Thm: Let X be geodesic then $\exists \delta$ st X is δ-hyperbolic $\iff$ $\exists \delta'$ s.t. it is (δ')-hyperbolic.

Pf: $\Rightarrow$: Assume WLOG $S \le M \le L$ are as above.

consider the comparison Δs



$M = d(z,y) + d(x,w) = a+b+c+d$

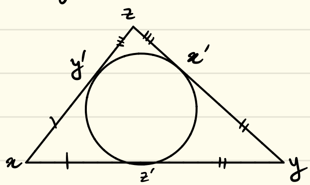
$d(z,w) = a + c - \ell$

$d(x,y) \le$ length of red path $\le b + \delta + \ell + \delta + d = b + d + \ell + 2\delta$

$\Rightarrow L = d(x,y) + d(z,w) \le a+b+c+d+2\delta = M + 2\delta$.

$\Leftarrow$: let us denote by x', y', z' the internal pts of $\Delta(x,y,z)$.

WTS $d(x',y') \le \delta$



Consider the 4 pts x, y, z, y'.

Out of the 3 opposite pairs the largest is $d(y,y') + d(x,z)$

since $d(y,y') \ge d(y,z') = d(y,x')$. (the other pairs are exactly perimeter/2)

Hence, $d(x,z) + d(y,y') \le d(y,z) + d(x,y') + 2\delta'$.

$$\left.\begin{aligned} &\Rightarrow d(y,y') \le d(y,x') + 2\delta' \\ &\text{Similarly } d(x,x') \le d(x,y') + 2\delta'. \end{aligned}\right\} (\ast\ast)$$

Now, consider the 4 pts x, y, x', y'.

The smallest of the 4 sums is clearly $d(x,y') + d(y,x')$.

So by the 4 pt ineq

$$d(x',y') + d(x,y) \le d(x,x') + d(y,y') + 2\delta'$$
$$\overset{(\ast\ast)}{\le} d(x,y') + d(y,x') + 6\delta' = d(x,y) + 6\delta' \Rightarrow d(x',y') \le 6\delta' =: \delta \;\square$$

Ex show that (δ)-hyperbolicity is not inv. under q.i. for general m.s. (hint: spiral $\overset{q.i.}{\simeq}$ line)

3. Approximating trees

Thm: X is geod. m.s. X is δ-hyperbolic iff $\forall n\ \exists C(n)$ such that $\forall x_1, \dots, x_n \in X$ $\exists$ a tree T embedded in X such that its edges are geodesics and $d(x_i, x_j) \leq d_T(x_i, x_j) + C(n)$. (where d_T in the distance measured in the metric tree)

Ex: Prove $\Leftarrow$ (hint: it is enough to consider 4 pts).

Def: a path $p:[a,b] \to X$ is τ-taut if $\text{length}(p) \leq d(p(a), p(b)) + \tau$ where

Obs: If p is τ-taut then it is a $(1,\tau)$-q. geodesic, and any subpath is τ-taut.
(write p in unit speed par. so that $\text{length}(p[s,t]) = t-s$,

Now,
$$\text{length}(p[a,s]) + \text{length}(p[s,t]) + \text{length}(p[t,b])$$
$$= \text{length}(p) \leq d(p(a), p(b)) + \tau$$
$$\leq d(p(a), p(s)) + d(p(s), p(t)) + d(p(t), p(b)) + \tau.$$

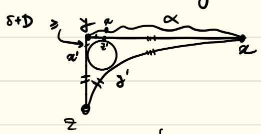
$d(p(s), p(t)) \leq \text{length}(p[s,t])$
$$\Rightarrow \quad \text{length}(p[s,t]) \leq d(p(s), p(t)) + \tau$$
$$\Rightarrow d(p(s), p(t)) \leq |t-s| \leq d(p(s), p(t)) + \tau$$

Lemma: Let X be δ-hyp geodesic. Let α be a τ-taut path btw $x, y \in X$. Then $\exists \tau'$ s.t. if y is the closest pt to some $z \in X$ on α then $\alpha \cup [y,z]$ is τ'-taut.

Pf: Recall that by stability of q. geod. $\exists D$ such that α is D-close to a geodesic $[x,y]$. Let δ' be such that $\text{insize}(\Delta) \leq \delta'$.

claim: $(x.z)_y \leq \delta' + D$.

since otherwise the internal pt $y' \in [x,z]$ is at distance $> \delta' + D$ from y. but $d(y', z') \leq \delta'$ and $\exists a \in \alpha$ s.t. $d(z', \alpha) \leq D$ so overall $d(z, \alpha) < d(z, y)$. ↯.



Thus
$$\text{length}(\alpha \cup [y,z]) = \text{length}\ \alpha + \text{length}\ [y,z]$$
$$\leq d(x,y) + \tau + d(y,z)$$
$$\leq d(x,z') + \delta + D + \tau + d(z, y') + \delta + D.$$
$$= d(x,y') + d(z,y') + 2(\delta + D) + \tau = d(x,z) + \overbrace{2(\delta+D) + \tau}^{\tau'} \quad \square$$

Pf of thm: $\Longrightarrow$ By induction on n we build a $\tau(n)$-taut tree. I.e a tree whose arcs are $\tau(n)$-taut.

If $n=2$, take $T_2 = [x_1, x_2]$. It is 0-taut.

For $n>2$, $x_1, \dots, x_n \in X$.

Assume we built T_{n-1} for $x_1, \dots, x_{n-1}$ which is $\tau(n-1)$-taut

let y be the closest pt to x_n in T_{n-1}.

let $T_n = T_{n-1} \cup [x_n, y]$. It is clear that T_n is an embedded tree and by the lemma it is $\tau_n = \tau'$-taut. $\square$