# Hyperbolic 3-manifolds

## Lecture 2: Hyperbolic Dehn fillings

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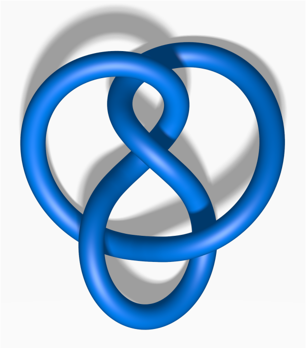
# The figure-eight knot complement

Let $K$ be the figure-eight knot[1]:



[1] jim.belk, from wikipedia

# The figure-eight knot complement – cont.

Consider the manifold $M = \mathbb{S}^3 - K$.

**Claim**

The manifold $M$ is homeomorphic to the following gluing of two tetrahedra.

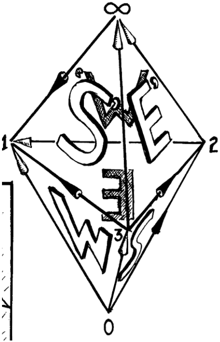
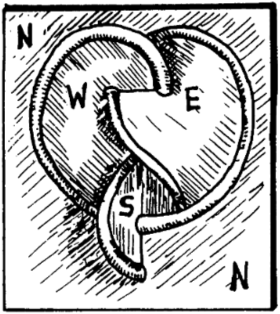



Figure: Picture by G. Francis, "A Topological Picturebook".

# The figure-eight knot complement – cont.

To see this, consider the following edges and disks:[2]

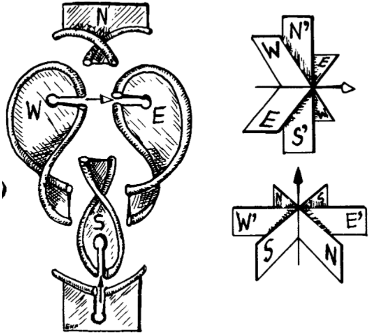



The two tetrahedra are the two 3-balls above and below the figure.

[2] Francis G., "A Topological Picturebook".

# The figure-eight knot complement – cont.

Around each edge we see the following configuration[3]:




**Exercise**

Find the gluing of the two tetrahedra from these figures.

[3] Francis G., “A Topological Picturebook”.

# The figure-8 knot complement – cont.

**Goal: Identify each tetrahedron with an ideal tetrahedron, and answer the following**

- Why does it have finite volume ?
- Why is it hyperbolic ? ?
- Why is it complete ? ? ?

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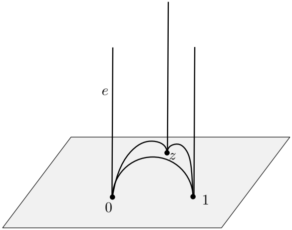
# The parameter of an ideal tetrahedron

Given an ideal tetrahedron $T$ and an edge $e$, up to an isometry of $\mathbb{H}^3$ we can place $T$ such that three of its vertices are $0, \infty, 1$, where the edge $e$ connects 0 and $\infty$, and the fourth ideal vertex is some $z(e) \in \mathbb{C}$ with $\Im z(e) > 0$.



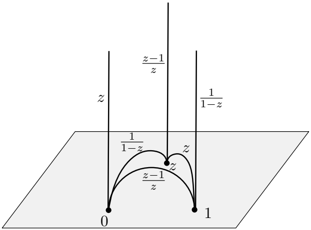

**Exercise**

Every ideal tetrahedron has finite volume.

# The parameter of an ideal tetrahedron – cont.

## Exercises

- $z(e)$ does not depend on the orientation of $e$.
- If $e, e'$ are opposite then $z(e) = z(e')$.
- If $z, z', z''$ are the invariants of the three edges incident to a vertex of $T$, in clockwise order, then $z' = \frac{z-1}{z}$ and $z'' = \frac{1}{1-z}$. In particular, they satisfy: $zz'z'' = -1$ and $1 - z - zz' = 0$.



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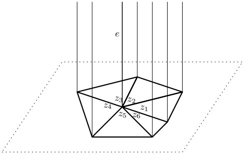
# The gluing condition

Assume $M$ is glued from ideal tetrahedra.Let $e$ be an edge of $M$, assume that $T_1, \dots, T_r$ are the tetrahedra of $M$ that are cyclically glued around $e$, and let $z_i$ be the invariant of the ideal tetrahedra $T_i$ at the edge $e$.

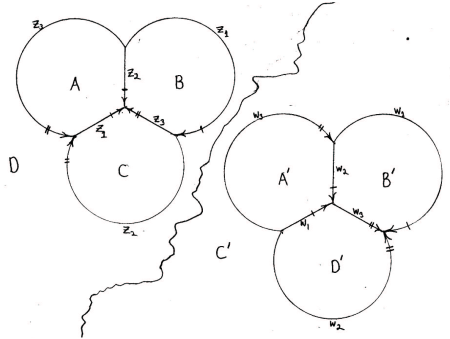



For $M$ to have a hyperbolic metric we need to verify that the dihedral angles around $e$ add up to $2\pi$. That is, $\arg(z_1) + \cdots + \arg(z_r) = 2\pi$.

But this does not suffice, we need to make sure that when going around an edge the accumulated gluing makes sense. This amounts to checking that $z_1 \dots z_r = 1$.

# Gluing condition for fig-8

Recall that the figure-eight knot complement $M = \mathbb{S}^3 - K$ is the gluing of two tetrahedra $M = T \cup T'$.Assume each is ideal, with parameters $z = z_1, z_2, z_3$ and $w = w_1, w_2, w_3$ as follows[4]:




$$z_1^2 z_2 w_1^2 w_2 = 1$$

$$z_2 z_3^2 w_2 w_3^2 = 1$$

$$\implies z(z-1)w(w-1) = 1$$

[4] Thurston W., "The Geometry and Topology of Three-Manifolds".

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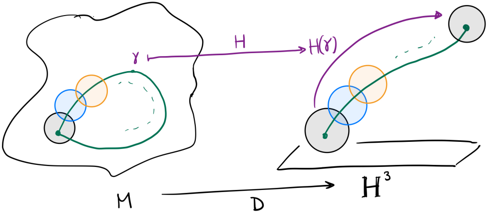
# The developing and holonomy maps

Given a hyperbolic metric $M$ we can consider its *developing map* $D : \widetilde{M} \to \mathbb{H}^3$ defined by sending a neighborhood of some point in $M$ to a neighborhood of a point in $\mathbb{H}^3$ and continuing the map along paths so that it is a local isometry.




The developing map is equivariant with respect to the *holonomy map* $H : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$.

## Exercise

Verify that $H$ is a well-defined homomorphism, and that the $D$ is well-defined and $H$-equivariant.

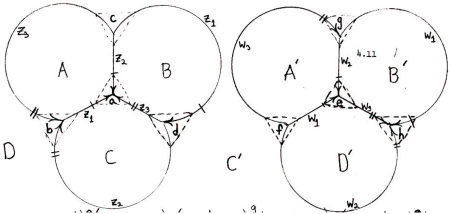
# Completeness condition

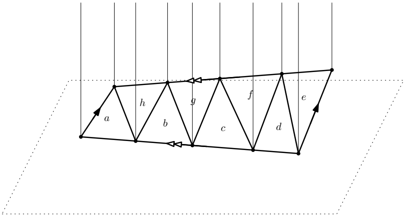
To check that the metric is complete, it suffices to check that a small neighborhood $N$ of the ideal vertex of $M$ is a cusp. This happens if and only if the holonomy image $H(\pi_1(N))$ of $\pi_1(N)$ is a discrete parabolic subgroup, that is, it lands in the subgroup $P = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ (up to conjugation).

Since $N$ is homeomorphic to a small neighborhood of $K$ in $\mathbb{S}^3 - K$ it is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, and $\pi_1(N) \simeq \mathbb{Z}^2$.

# Figure-eight knot complement

Explicitly, the developing map of $N$ given as follows[5]:



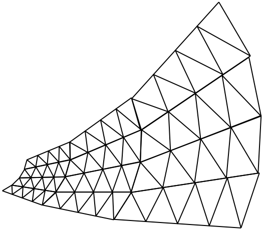




We see that the holonomy image $H(\pi_1(N))$ fixes the point $\infty$,i.e. it is in the subgroup $B = \left\{ \begin{pmatrix} z & * \\ 0 & z^{-1} \end{pmatrix} \right\}$. The group $B$ acts on $\mathbb{C}^2$ by affine linear transformation $\{z \mapsto az + b\}$. It lands in $P$ if and only if the affine transformations are translations (i.e. $a = 1$).

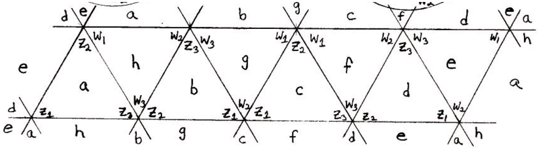
[5] Thurston, "The Geometry and Topology of Three-Manifolds"

# The developing map around an ideal vertex

If it does not land in $P$ the developing map will look like:



Let us compute the linear part (the derivative) of the holonomy:




$H'(x) = \left(\frac{z}{w}\right)^2$ and $H'(y) = w(1-z)$.

# Completeness for the figure-eight knot complement

To summarize, for the gluing of tetrahedra to be hyperbolic and complete we need to satisfy:

| | | |
|---|---|---|
| hyperbolic: | $z(z-1)w(w-1)=1$ | $\Im(z)>0, \Im(w)>0$ |
| complete: | $\left(\frac{z}{w}\right)^2 = H'(x) = 1$ | $w(1-z) = H'(y) = 1$ |

We see that $z_0 = w_0 = e^{2\pi i/3}$ is the only solution to all equations. Note that this is exactly the parameter of the regular ideal tetrahedron.

This completes the proof that the Figure-eight knot complement has a finite-volume hyperbolic metric.

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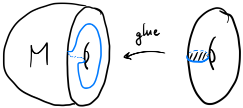
# Dehn Fillings

## Definition

Let $M$ be a compact manifold with tori boundaries. $M'$ is a *Dehn filling* of $M$ if it is obtained from $M$ by gluing solid tori to (some of) its boundary components.




## Theorem (Lickorish – Wallace)

*Any closed orientable 3-manifold can be obtained by Dehn filling a link complement.*

**Goal:** Hyperbolic Dehn fillings of the figure-eight knot complement.

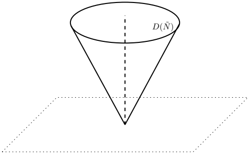
# Incomplete metrics

Let us analyze further the case of incomplete metrics on $M$.That is, $H'(x), H'(y)$ are not both 1

Both $H(x), H(y)$ fix $\infty$ and since they are commuting contracting (or expanding) maps they both fix one point in $\mathbb{C}^2$. WLOG, let this point be 0.

So the holonomy simply consists of linear maps ($z \mapsto az$), and is determined by $H(\cdot)' : \pi_1(N) \to \mathbb{C}^*$. This map can be lifted to $\widetilde{H} : \pi_1(N) \to \widetilde{\mathbb{C}^*} \simeq \mathbb{C}^2$ (where the last map is log map).

It stabilizes the geodesic $\eta$ connecting $0, \infty$ (the $z$-axis), and the action of $g \in \pi_1(N)$ on $\eta$ is given by $H(g).t = |H'(g)|t$. Finally, the developing image $D(\tilde{N}) = C - \eta$ where $C$ is, WLOG, the cone around the z-axis $\eta$:



# Completion of incomplete metrics

If there are generators $a, b$ for $\pi_1(N) \simeq \mathbb{Z}^2$ so that $\widetilde{H}(a) = \pm 2\pi i$. Then $\pi_1(N)$ acts discretely and freely on $C$.

The completion of $C - \eta$ is the full cone $C$. By moding out by $\pi_1(N)$ we get that the completion $\bar{N}$ of $N$ is the hyperbolic solid torus $C/N$ (obtained by adding a circle $\eta/N$ to $N$).

Thus, the completion $\bar{M}$ of $M$ is a Dehn filling. Note also that the curve described by $a$ is the slope of the filling.

# Hyperbolic Dehn filling

Given $\widetilde{H}(x), \widetilde{H}(y)$ there are unique $\alpha, \beta \in \mathbb{R}$ (up to sign) so that

$$\alpha \widetilde{H}(x) + \beta \widetilde{H}(b) = \pm 2\pi i.$$

Thus, to find hyperbolic Dehn fillings it suffices to find a **primitive pair of integers** $(\alpha, \beta) \in \mathbb{Z}^2$, because in this case $a = \alpha x + \beta y$ is part of a generating set $a, b$ for $\pi_1(N)$ which has the desired properties.

# Hyperbolic Dehn filling – cont.

Summary, for the figure-eight knot complement:Given the parameter $w$ for one of the ideal tetrahedra we get

- $\rightsquigarrow z$
- $\rightsquigarrow \widetilde{H}(x), \widetilde{H}(y)$
- $\rightsquigarrow (\alpha, \beta)$.

So we get a map $S : w \mapsto \alpha + i\beta \in \mathbb{C}$, we can extend it to $w_0$ by sending $S(w_0) = \infty \in \bar{\mathbb{C}}$.

This map $S : U \to \bar{\mathbb{C}}$ is analytic in the neighborhood of $w_0$,and so $S(U)$ is a neighborhood of $\infty$ in $\mathbb{C}$.Therefore, for all but finite many primitive pairs $(\alpha, \beta) \in \mathbb{Z}^2$ we have $\alpha + i\beta \in S(U)$.

# Hyperbolic Dehn Filling Theorem

**Theorem (Thurston's Hyperbolic Dehn Filling)**

*Let $M$ be a compact manifold with tori boundaries $\partial M = T_1 \cup \ldots \cup T_r$. If $M - \partial M$ admits a complete (finite-volume) hyperbolic metric then with the exception of finitely many slopes for each $1 \leq i \leq r$, the Dehn filling of $M'$ is hyperbolic.*