Hyperbolic 3-manifolds

Lecture 3: Mostow Rigidity

Nir Lazarovich

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Mostow Rigidity Theorem

We saw that in dimension 2 there are many ($6g-6$ dimensional) different hyperbolic metrics on a given closed surface of genus $g \geq 2$.

Theorem (Mostow-Prasad Rigidity Theorem)

Let $n \geq 3$, and let M, M' be finite volume complete hyperbolic n-manifolds, then any homotopy equivalence $f : M \to M'$ is (homotopic) to an isometry $M \to M'$.

Corollary

Let $n \geq 3$, and let M, M' be finite volume complete hyperbolic n-manifolds, then any isomorphism $f : \pi_1(M) \to \pi_1(M')$ is induced from an isometry $M \to M'$.

Corollary

Any isomorphism between (torsion-free) lattices in $\mathrm{PSL}_2(\mathbb{C})$ comes from a conjugation in $\mathrm{PSL}_2(\mathbb{C})$.

Lifting to $\mathbb{H}^3$

We will prove the theorem for closed hyperbolic 3-manifolds.

Let $f : M \to M'$ be the homotopy equivalence, and let $f' : M' \to M$ be its homotopy inverse (i.e. $f' \circ f \simeq \mathrm{id}_M$, and $f' \circ f \simeq \mathrm{id}_{M'}$). Up to a small homotopy, we may assume that both f, f' are Lipschitz.

The map F lifts to an (f_*-)equivariant Lipschitz map between the universal covers
$$F : \widetilde{M} = \mathbb{H}^3 \to \mathbb{H}^3 = \widetilde{M}'.$$

Quasi-isometry

Lemma

The map F is a quasi-isometry:there exists $L > 1, C \geq 0$ such that for all $x, y \in \mathbb{H}^3$, $\quad L^{-1}d(x,y) - C \leq d(F(x), F(y)) \leq Ld(x,y) + C$.

Proof.

The right inequality follows because F is Lipschitz.
The homotopy H between id_M and $f' \circ f$ lifts to a homotopy $\widetilde{H}$ between $\mathrm{id}_{\mathbb{H}^3}$ and $F' \circ F$.
In particular, since M is compact, there is some C such that $d(x, F' \circ F(x)) \leq D$ for all $x \in \mathbb{H}^3$.
Therefore,

$$d(x,y) \leq d(F' \circ F(x), F' \circ F(y)) + 2D \leq Ld(F(x), F(y)) + 2D$$

and the inequality follows. □

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The Morse Lemma

Lemma (The Morse Lemma)

There exist R such that for every geodesic segment γ in $\mathbb{H}^3$ there exists a geodesic γ' at (Hausdorff) distance R from $F(\gamma)$.

Corollary (Boundary map)

The map $F : \mathbb{H}^3 \to \mathbb{H}^3$ defines a f_*-equivariant map $\partial F : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$.

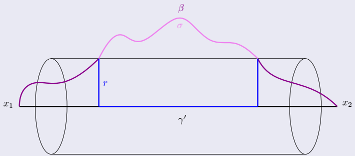
Proof of the Corollary.

Let $x \in \partial\mathbb{H}^3$,let γ be any geodesic ray ending in x. Then $F(\gamma)$ stays a bounded distance from some geodesic ray η. The corresponding endpoint of η is $\partial F(x)$. This is well-defined since any two geodesics ending in x are at bounded distance, and so are their images. □

Morse Lemma – cont.

Proof.

Assume first that γ is finite. Let $\beta = F(\gamma)$, and Let γ' be the geodesic segment connecting its endpoints.



Let $N_r(\gamma')$ be the r neighborhood of γ'. Then, if σ is a subsegment of β outside $N_r(\eta)$ with endpoints on $\partial N_r(\gamma')$ then its projection to γ' has length $\leq \text{length}(\sigma)/\cosh(r)$.

It follows that the subsegments of β outside η are bounded, as otherwise it would be more efficient to travel via η (at length $\leq 2r + \text{length}(\sigma)/\cosh(r)$) contradicting the quasi-isometry inequality.

By enlarging r, there exists R such that $\beta \subset N_r(\eta)$.

If β is infinite just take subsegments of β converging to β and get the desired geodesic as the limits of the corresponding γ'. □

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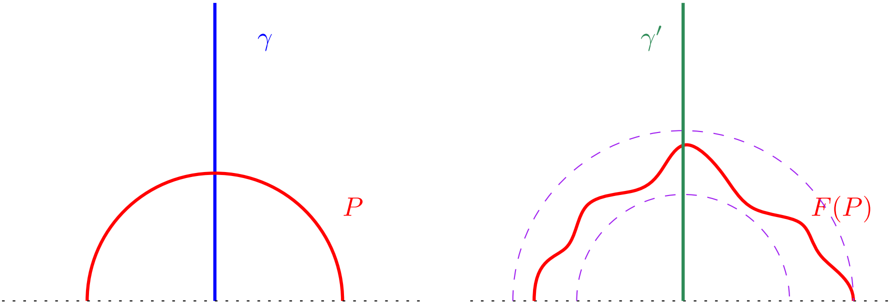
Next we would like to prove that the map ∂F is continuous.

Lemma (More Morse)

There exists C such that for every geodesic line γ, and perpedicular plane P to γ. The projection of $F(P)$ to the geodesic γ' (from the Morse Lemma) has bounded diameter.

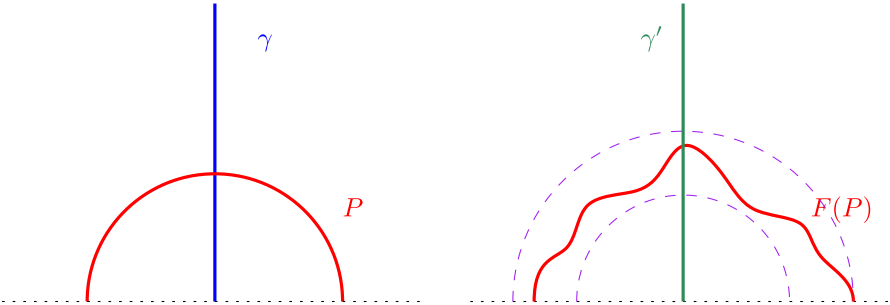


Proof

Continuity

Lemma

The map ∂F is continuous.



Quasi-conformal mappings

Definition

A homeomorphism $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is *quasi-conformal* if there exists K such that for all $z \in \bar{\mathbb{C}}$

$$\lim_{r \to 0} \frac{\sup \{d(h(x), h(y))\}}{\inf \{d(h(x), h(y))\}} \leq K$$

where x, y run over all antipodal points on the sphere of radius r around z.

Remark

The map h is conformal if and only if $K = 1$. Every conformal homeomorphisms of $\bar{\mathbb{C}}$ is a M obius transformation.

Exercise

Show that ∂F is quasi-conformal.

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The proof of Mostow rigidity

Summary, we saw that the map $f : M \to M'$ induces a map $F : \mathbb{H}^3 \to \mathbb{H}^3$ which is equivariant, and a quasi-conformal map $\partial F : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ which is equivariant with respect to $f_* : \pi_1(M) =: \Gamma \to \Gamma' := \pi_1(M')$.

Theorem (Bers)

Every quasi-conformal map $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a.e differentiable (as a real function).

At almost every point one can consider the derivative of ∂F.

Proof of Mostow rigidity – cont.

Theorem (Double ergodicity on the boundary)

The diagonal action $\Gamma \curvearrowright (\bar{\mathbb{C}})^2$ is ergodic. In particular, $\Gamma \curvearrowright \bar{\mathbb{C}}$ is ergodic.

Proof.

To every pair of distinct points in $(\bar{\mathbb{C}})^2$ there is a geodesic in $\mathbb{H}^3$ connecting them. So, up to measure zero, $(\bar{\mathbb{C}})^2 = \mathrm{PSL}_2(\mathbb{C})/A$ where $A = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right\}$.The action $\Gamma \curvearrowright \mathrm{PSL}_2(\mathbb{C})/A$ is ergodic if and only if the action $\Gamma \backslash \mathrm{PSL}_2(\mathbb{C}) \curvearrowleft A$ is ergodic.But this is exactly the ergodicity of the geodesic flow! □

The set of points in which ∂F is conformal is Γ-invariant measurable subset of $\bar{\mathbb{C}}$. By ergodicity, ∂F is either a.e. conformal or a.e. non-conformal.

Proof of Mostow rigidity – cont.

If the map ∂F is a.e. non-conformal, then at a.e point we can consider the line field of the direction that is stretched the most by the derivative $d\partial F$. This is a Γ-invariant measurable line field on $\bar{\mathbb{C}}$.

For a.e two points $x, y \in \bar{\mathbb{C}}$, consider the two circles that pass through x, y and are tangent to the line fields at x and y respectively. The angle between them is a Γ-invariant measurable function on $(\bar{\mathbb{C}})^2$.By double ergodicity, this function is constant, but this is not possible.

Finally, if ∂F is conformal, then it is an element of $g \in \mathrm{PSL}_2(\mathbb{C})$. This element conjugates the action of Γ and Γ'. completing the proof.