Hyperbolic 3-manifolds

Lecture 4: Hyperbolization, Geometrization and knots

Nir Lazarovich

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Thurston's Hyperbolization Theorem

Theorem (Thurston's Hyperbolization Theorem)

If M is a closed (orientable) irreducible atoroidal Haken 3-manifold then M has a complete hyperbolic metric.

Remarks

- There is a version of this theorem for manifolds with boundary.
- As a corollary of Perelman's Geometrization Theorem the assumption that M is Haken can be replaced by $\pi_1(M)$ is infinite.

Definitions

Definition

- A manifold M is *irreducible* if every $\mathbb{S}^2$ bounds a 3-ball.
- A closed manifold M is *atoroidal* if there is no embedded torus which is π_1-injective.
- A closed orientable irreducible 3-manifold M is *Haken* if it contains an embedded orientable surface $S \subseteq M$ (which is not a sphere) and is π_1-injective.

Haken Hierarchy

Ingredients of the proof of Hyperbolization:

Strategy: Haken Hierarchy.

Fact

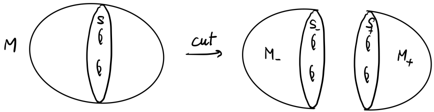
If M has non trivial homology then M is Haken, in particular, if M has boundary (and it is not a 3-ball) then M is Haken.

If M contains such an incompressible surface then M can be cut along this surface to obtain a manifold with boundary, continue cutting until M reduces to a union of 3-balls. This process is a *Haken hierarchy* for M.

Idea: Prove that M has a complete hyperbolic structure by induction on the depth of the hierarchy.

The inductive step setup and strategy

Inductive step:
Cut M along S to obtain M'. There are two copies of S, $S_\pm \subset \partial M'$, let $M_\pm$ be the corresponding connected components of M' (possibly $M_+ = M_-$).



By the induction hypothesis $M' - \partial M'$ has a complete hyperbolic metric.

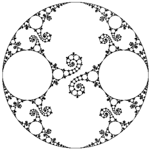
Strategy: (1) Glue the manifolds M_+, M_- along S_+, S_- after (2) deforming the given metric on M_+ and M_- so that (3) the metric on S_+, S_- agree.

E.g. if the surfaces S_-, S_+ are geodesic, and the identification $S_- = S_+$ is an isometry, then M_-, M_+ can be glued.

Limit sets and Ahlfors' Finiteness

By the induction hypothesis, for each component C of M' there is a discrete subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ with $\Gamma \simeq \pi_1(C)$ such that $\mathbb{H}^3/\Gamma \cong C$.

The **limit set** $\Lambda(\Gamma)$ is the set of accumulation points on $\partial\mathbb{H}^3$ of an orbit $\Gamma.o$ of $o \in \mathbb{H}^3$. We assume that $\Lambda(\Gamma)$ is connected. [1]



Let $\Omega = \partial\mathbb{H}^3 - \Lambda(\Gamma)$ be the *domain of discontinuity* of Γ.

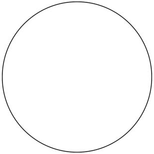
Theorem (Ahlfors' finiteness theorem)

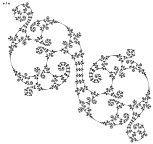
If $\Gamma = \pi(M')$ is finitely generated Kleinian group, then Ω/Γ is a collection of finitely many surfaces, each of which homeomorphic to a boundary component of M'.

[1] Picture by Curt McMullen

(1) Gluing: Maskit's combination theorem

Let $\Sigma_\pm \leq \Gamma_\pm$ be the subgroups corresponding to $\pi_1(S_\pm) \leq \pi_1(M_\pm)$, and assume that they are *quasi-Fuchsian* – i.e. their limit set is a Jordan curve.[2]





Theorem (Maskit's Combination Theorem)

Assume further that $\Gamma_+ \cap \Gamma_- = \Sigma = \Sigma_\pm$, *and that* $\Omega_\pm$ *are components of* $\partial\mathbb{H}^3 - \Lambda\Sigma$ *and that they are precisely invariant under* Σ *in* $\Gamma_\pm$ *respectively. Then* $\langle \Gamma_+, \Gamma_- \rangle = \Gamma_+ *_\Sigma \Gamma_-$ *is a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$.

A similar theorem exists for the HNN case

[2] Chris King, https://dhushara.com/DarkHeart/quasif/quasi.htm

(2) Deforming: Bers' Deformation Theory

Given a discrete torsion-free subgroup $\Gamma \leq \mathsf{PSL}_2(\mathbb{C})$ and $M = \mathbb{H}^3/\Gamma$.
Assume $\Lambda\Gamma$ is connected.

Theorem (Bers' Isomorphism)

There is an identification between deformations (by quasi-conformal mapping) of Γ and the Teichmüller space of all complex structures on Ω/Γ.

Note that in particular, if S is a surface, $\Gamma = \pi_1(S)$ is quasi-Fuchsian, then $\Lambda\Gamma$ is a Jordan curve and Ω is a pair of open disks. By Bers' Isomorphism the group Γ is described by a pair of complex structure on S.
(Simultaneous Uniformization)

(3) Matching: the Skinning map

Case 1. $M' \ncong S \times [0,1]$**:**
By the induction hypothesis $M' - \partial M'$ has a complete hyperbolic metric, moreover, each of $S_\pm$ is quasi-Fuchsian.
Let us define the *skinning map* σ: A complex structure X_+ on S_+ determines a hyperbolic structure on M_+. The hyperbolic metric on M_+ determines a pair of complex structures on S_+ one of which is X_+ and the other is denoted $\sigma(X_+)$.
We want to find complex structures (X_+, X_-) so that $(X_-, X_+) = \tau\sigma(X_-, X_+)$, where τ is the identification switching S_+ and S_-.

Theorem (Thurston)

If $M' \ncong S \times [0,1]$*, the map* $\tau \circ \sigma$ *has a fixed point.*

The fibered manifold case

Case 2. $M' = S \times [0,1]$**:**
In this case M is a fibered 3-manifold, and it is given by $M = S \times [0,1]/(x,1) \sim (\phi(x),0)$ for some homeomorphism ϕ of S. $\pi_1(M) = \pi_1(S) \rtimes \mathbb{Z} = \langle \pi_1(S), t | tht^{-1} = \phi_*(h) \forall h \in \pi_1(S) \rangle$. Moreover, since M is atoroidal, the homeomorphism is of a special kind called *pseudo-Anosov*.

Goal: find a discrete faithful representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. To do so, we find a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ and show that there is some $g \in \mathrm{PSL}_2(\mathbb{C})$ such that $g\rho g^{-1} = \rho\phi_*$.

Theorem (Thurston)

The map ϕ_* has a fixed point in the space of discrete faithful representations of $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$

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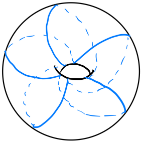
Hyperbolic knots

Corollary

If a knot K is not a torus knot and not a satellite knot, then $\mathbb{S}^3 - K$ has a complete hyperbolic metric of finite volume.

- A *Torus knot* is a closed curve on the standard torus in $\mathbb{S}^3$:



- A *satellite knot* is a knot that can be constructed as follows:

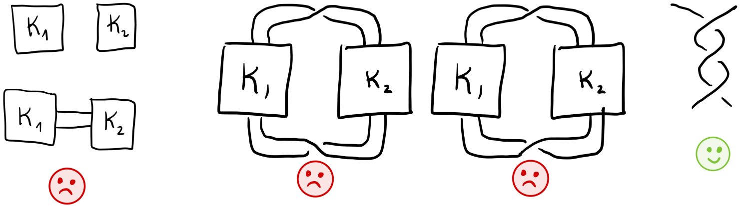
Detecting hyperbolic knots from diagrams

Theorem (Menasco)

Let K be a prime alternating knot, then either K is a torus knot or K is hyperbolic.

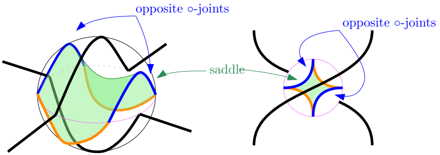
Theorem (L.-Moriah-Pinsky, Futer-Purcell for 6-highly-twisted)

If K has a connected, prime, twist-reduced, 3-highly-twisted knot diagram (with at least two twists regions) then K is hyperbolic.



Proof

Let K be a knot as in the theorem.
Goal: show that $M = \mathbb{S}^3 - K$ is atoroidal.
Place the knot K on the projection plane P, so that at every crossing you put a "bubble". The torus intersects each bubble in a stack of saddles:



Let P_+ (resp. P_-) be the plane P with the upper (lower) hemispheres of the bubbles. Consider the curves of intersection of $T \cap P_\pm$.

Proof – cont.

One can compute the Euler characteristic of T from the curves of intersection with $P_\pm$ by:

$$0 = \chi(T) = \sum_{c \subset T \cap P_\pm} \chi_+(c)$$

where $\chi_+(c) = (1 - \frac{1}{4}\#\{\text{bubbles c meets}\})$.
For some curves $\chi_+(c) > 0$.

Idea: Define $\chi'(c)$ in such a way that the positive contributions are distributed among other curves so that now:

$$0 = \chi(T) = \sum_{c \subset T \cap P_\pm} \chi'(c) \quad \text{and} \quad \chi'(c) \leq 0.$$

It follow that $\chi'(c) = 0$ for all curves, a case by case analysis shows that T must be parallel to the knot.

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JSJ and Geometrization

Theorem (JSJ decomposition, Jaco-Shalen, Johannson)

Every closed irreducible 3-manifold M has a disjoint collection of tori $T_1, \ldots, T_n$ such that the components of $M - \cup_i T_i$ are atoroidal or SFS (Seifert fibered spaces).

Theorem (Perelman, Thurston for Haken manifolds)

Every closed irreducible 3-manifold M has a disjoint collection of tori $T_1, \ldots, T_n$ such that each component of $M - \cup_i T_i$ has a finite volume complete structure of one of the 8 Thurston geometries:

$$\mathbb{S}^3, \quad \mathbb{E}^3, \quad \mathbb{H}^3, \quad \mathbb{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \widetilde{\mathsf{PSL}_2(\mathbb{R})}, \quad Nil, \quad Sol.$$

Thank You!

and...

Thanks to the organizers for the great workshop!