Hyperbolic 3-manifolds

Lecture 1: hyperbolic geometry

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Theorem (Killing–Hopf)

For every $n \geq 2$ and $\kappa \in \mathbb{R}$ there is exactly one simply connected, complete Riemannian manifold of dimension n with constant sectional curvature κ. Up to rescaling, they are:

- $\kappa = 0$, $\mathbb{E}^n =$ *Euclidean space.*
- $\kappa = 1$, $\mathbb{S}^n =$ *Spherical space.*
- $\kappa = -1$, $\mathbb{H}^n =$ *Hyperbolic space.*

The *hyperbolic n-space* $\mathbb{H}^n$ can be defined as follows:

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

equipped with the metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

Hyperbolic geometry in dimension 2

In dimension 2, we can identify $\mathbb{H}^2$ with the set $\{z = x_1 + x_2 \mathrm{i} \in \mathbb{C} \mid \Im(z) = x_2 > 0\}$.
The group $\mathrm{PSL}_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\} \curvearrowright \mathbb{H}^2$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}$$

Exercise

This action preserves the *hyperbolic metric*

$$ds^2 = \frac{dx_1^2 + dx_2^2}{\Im(z)^2}$$

and it is the full group of orientation preserving isometries of $\mathbb{H}^2$.

Hyperbolic geometry in dimension 3

In dimension 3, one can consider the quaternions
$\mathrm{H} = \{z = x_1 1 + x_2 \mathrm{i} + x_3 \mathrm{j} + x_4 \mathrm{k} \mid a, b, c, d \in \mathbb{R}\}$, and consider the set

$$\mathbb{H}^3 = \{x_1 1 + x_2 \mathrm{i} + x_3 \mathrm{j} \mid x_3 > 0\}.$$

Now, the group $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\} \curvearrowright \mathbb{H}^3$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = (az + b)(cz + d)^{-1}.$$

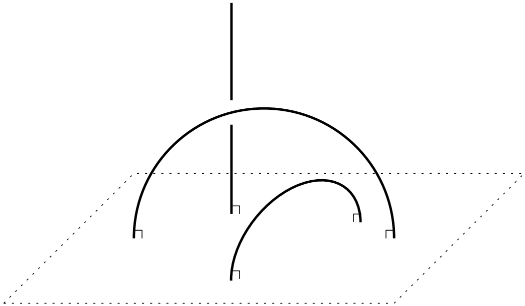
$\mathrm{PSL}_2(\mathbb{C})$ is the group of orientation preserving isometries of the metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.$$

Let us denote $\partial \mathbb{H}^3 = \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Geodesics in $\mathbb{H}^3$

Geodesics in $\mathbb{H}^n$ are circular arcs (and lines) which are perpendicular to $\partial\mathbb{H}^3$.



Isometries of $\mathbb{H}^3$

By Jordan's Theorem, each matrix in $\mathrm{PSL}_2(\mathbb{C})$ is conjugate to:

- *hyperbolic* $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $|a| > 1$.
- *elliptic* $\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$ with $|a| = 1$.
- *parabolic* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Exercise

How do they act on $\mathbb{H}^3$ and on $\partial\mathbb{H}^3$?

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Hyperbolic manifold

We will say that an oriented Riemannian manifold M is hyperbolic if it has sectional curvature -1 at every point.

If M is a complete hyperbolic manifold then $\widetilde{M} \simeq \mathbb{H}^n$. The action of M on its universal cover by deck transformations gives an isomorphism $\pi_1(M) \to \Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ and $M \simeq \mathbb{H}^n/\Gamma$.
Discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ are called *Kleinian groups*.

Exercise

Any finite subgroup F of $\mathrm{PSL}_2(\mathbb{C})$ fixes a point in $\mathbb{H}^3$. [Hint: show that F preserves some inner-product in $\mathbb{C}^2$.]

Therefore if M is a complete hyperbolic 3-manifold then $\pi_1(M)$ is torsion free.
Conversely, every discrete torsion-free subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$ is the fundamental group of the complete hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$.

We will mostly care about finite volume (or even closed) hyperbolic 3-manifolds.

Definition

A *lattice* $\Gamma \leq G$ is a discrete subgroup of finite co-volume (i.e. G/Γ has a finite G-invariant Radon measure). A lattice is *uniform* if G/Γ is compact.

Fact

The manifold M has finite volume (resp. closed) if and only if $\Gamma = \pi_1(M)$ is a lattice (resp. uniform lattice) in $PSL_2(\mathbb{C})$.

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Algebraic constructions

Theorem (Borel – Harish-Chandra)

Let $\mathbb{G}$ be an algebraic group defined over $\mathbb{Q}$ without $\mathbb{Q}$ characters, then $\mathbb{G}(\mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$. In particular, if there exists an epimorphism $f : \mathbb{G}(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{C})$ with compact kernel then $f(\mathbb{G}(\mathbb{Z}))$ is a lattice in $\mathrm{PSL}_2(\mathbb{C})$.

Examples

1. $\mathrm{PSL}_2(\mathbb{Z}[i])$ is a lattice in $\mathrm{PSL}_2(\mathbb{C})$.
2. $\mathrm{PSL}_2(\mathbb{Z}[\omega])$ where $\omega = e^{2\pi i/3}$ is a lattice in $\mathrm{PSL}_2(\mathbb{C})$.
3. Bianchi groups: Let $d > 0$ be a square-free integer, then $\mathrm{PSL}_2(\mathcal{O}_d)$ is a lattice in $\mathrm{PSL}_2(\mathbb{C})$, where $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}[\sqrt{-d}]$.
4. $\mathbb{G} = \mathrm{SO}(x_1^2 + x_2^2 + x_3^2 - \sqrt{2}x_4^2)$ can be defined over $\mathbb{Q}$ so that $\mathbb{G}(\mathbb{R}) \simeq \mathrm{SO}(3,1) \times \mathrm{SO}(4)$. Now consider the map $\mathbb{G}(\mathbb{R}) \to \mathrm{SO}(3,1)$ and note that $\mathrm{SO}(3,1) \simeq \mathrm{SL}_2(\mathbb{C})$.

Algebraic constructions – cont.

To get a torsion free lattice we can use the following:

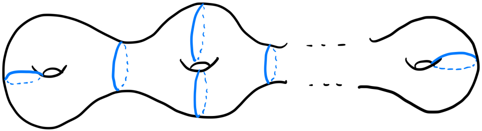
Theorem (Selberg)

Every finitely generated linear group (over a field with characteristic zero) has a finite index torsion-free subgroup.

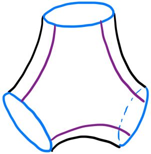
This is cheating... We want to start with a manifold and "discover" a hyperbolic metric.

Closed hyperbolic surfaces

Let's start in dimension 2. Let S_g be the closed surface of genus $g \geq 2$. We can cut the surface along $3g - 3$ curves into pairs of pants.

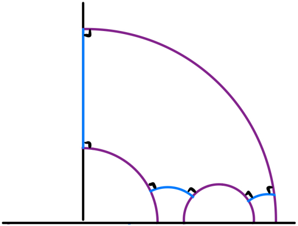


Each of the pairs of pants we cut along the 3 seams to obtain two hexagons.



Closed hyperbolic surfaces – cont.

Each hexagon can be realized as a right-angled hyperbolic hexagon (with geodesic edges).



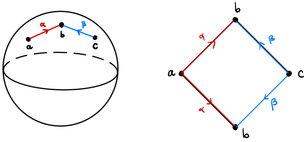
Finally, gluing them back together, one obtains a hyperbolic closed surface.

Moreover, each such surface is given by the lengths of the $3g-3$ curves, and additional $3g-3$ twist parameters.

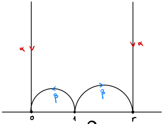
These are the $6g-6$ *Fenchel-Nielsen* coordinates for the moduli space of all hyperbolic metrics on a surface of genus g.

Finite volume hyperbolic surfaces

Let $\Sigma = \mathbb{S}^2 - \{a, b, c\}$ be the thrice punctured sphere.Cut Σ along arcs α, β connecting a, b to c to obtain a quadrilateral whose vertices are removed:



Realize the quadrilateral as a hyperbolic quadrilateral with 'ideal' vertices:



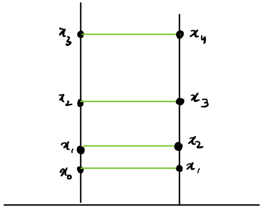
Now glue them using isometries of $\mathbb{H}^2$, to obtain a hyperbolic metric on Σ. There are 3 degrees of freedom in this construction (1 for the choice of the of the ideal quadrilateral, and 2 for the choice of the isometries).

Finite volume hyperbolic surfaces – cont.

But, as some of you might know, the moduli space of Σ is a point.

Oops!

Some of the hyperbolic metrics we got are NOT COMPLETE!!!



Cusps

Exercise

Check that if all the gluing maps around a, b and $c(!)$ are parabolic then the metric is complete.

The neighborhood of the points a, b, c in the complete hyperbolic metric are *cusps*:

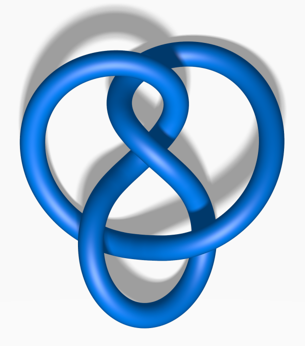
Definition

A *cusp* of M is a submanifold that is isometric to a neighborhood of $\infty \in \partial\mathbb{H}^n$ in the quotient $\mathbb{H}^n/\Gamma$ for some discrete paraboloic subgroup $\Gamma \leq \mathbb{H}^n$ which acts cocompactly on $\partial\mathbb{H}^n - \{\infty\}$.

Fact

Every complete finite-volume hyperbolic manifold M has finitely many cusps $K_1, \dots, K_r$ such that $M - \bigcup K_i$ is compact.

Let K be the figure-eight knot[1]:



[1] jim.belk, from wikipedia

The figure-eight knot complement – cont.

Consider the manifold $M = \mathbb{S}^3 - K$.

Claim

The manifold M is homeomorphic to the following gluing of two tetrahedra.

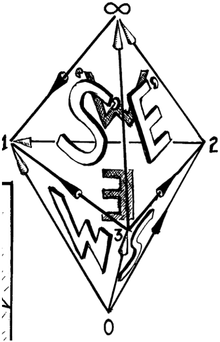
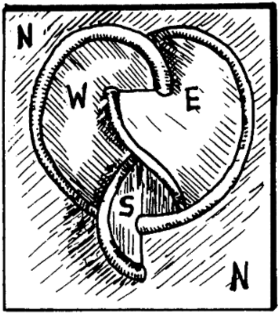


Figure: Picture by G. Francis, "A Topological Picturebook".

The figure-eight knot complement – cont.

To see this, consider the following edges and disks:[2]

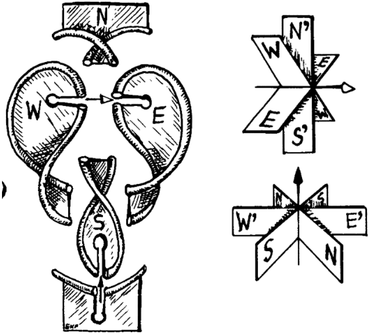


The two tetrahedra are the two 3-balls above and below the figure.

[2] Francis G., "A Topological Picturebook".

The figure-eight knot complement – cont.

Around each edge we see the following configuration[3]:



Exercise

Find the gluing of the two tetrahedra from these figures.

[3] Francis G., "A Topological Picturebook".

The figure-eight knot complement – cont.

Now, we can identify each tetrahedron with the regular ideal tetrahedron, i.e. the tetrahedron whose vertices are ∞ and the vertices of an equilateral triangle in $\mathbb{C}$:

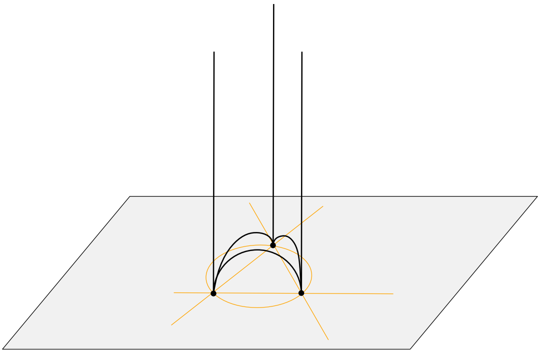


Figure: The regular ideal tetrahedron

The figure-8 knot complement – cont.

We claim that this gives a complete finite volume hyperbolic metric on M.

- Why does it have finite volume ?
- Why is it hyperbolic ? ?
- Why is it complete ? ? ?

To be continued...